DUAL FORMS AND THE LONDON EQUATIONS

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DUAL FORMS AND THE LONDON EQUATIONS

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CHAPTER 1

DIFFERENTIABLE MANIFOLDS AND VECTOR FIELDS

Introduction

The basis of this dissertation is to outline two important concepts of modern mathematics; namely, the idea of a differentiable manifold and differential forms on a manifold. In this context, classical vector analysis will be re-formulated and cast in the language of differential forms on a 3-manifold embedded in Euclidean 3-space.

The advantages of doing this are two-fold:

(i) Many of the results of classical vector analysis and electromagnetic theory are more succinctly expressed in this way and are more amenable to generalization. A typical example is Stokes' Theorem which actually can be generalized to an n-manifold.

(ii) Having cast the problems of electromagnetic theory in the context of 4-manifolds in space-time, their solution becomes amenable to the methods of such powerful topics as :

Lie Groups
Differential Geometry
Algebraic Topology,

but, since each of these topics are vast in their ramifications, they cannot be entered into in any depth here. Indeed, only the last of these topics will receive mention.

Chapter 1 describes the underlying concept of differentiable manifolds and vector fields on a manifold.

Chapter 2 begins by defining tensors and tensor product on a vector space and it then goes on to define tensors and forms on a differentiable manifold. A very important idea is that of a dual 1-form of a C^∞ vector field on a manifold. It is the device which facilitates the description of vector analysis in terms of differential forms.

The specific aim of this dissertation is to show how these ideas can be applied to electromagnetic theory in general (eg. Maxwell's Equations) and to superconductivity in particular. Chapter 3 outlines how this is done and re-writes the London Equations in terms of differential forms.

1.2 MANIFOLDS

There is a conceptual gap between the idea of a curve or a surface in Euclidean space and the idea of a topological or differentiable manifold. This gap is bridged by use of some of the notions and techniques of the calculus of functions of the type:

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m.$$

For example, the idea of the differential of such a function, f, at a point $\underline{a}$ in its domain as being a linear map from the tangent space

$$T_{\underline{a}}(\mathbb{R}^n) \text{ to } T_{f(\underline{a})}(\mathbb{R}^m)$$

will be used and generalized.

To cover the basic groundwork of a course on differentiable manifolds, one would refer to the following theorems from advanced calculus:

CHAIN RULE This generalizes to a similar chain rule for C^∞ maps between manifolds.

INVERSE FUNCTION THEOREM An analogous version of this for a C^∞ map between manifolds helps in the construction of local diffeomorphisms.

IMPLICIT FUNCTION THEOREM When generalized to certain types of functions between manifolds, one is able to define submanifolds of the given manifold (differentiable varieties in particular).

The context of this dissertation is such that only the first of these results will be referred to.

1.2.1 Calculus of Vector Functions

Let

$$\mathbb{R}^n = \{(a_1,\ldots,a_n) : a_i \in \mathbb{R}\}$$

For i = 1,2,...,n, let

$$u_i : \mathbb{R}^n \to \mathbb{R}$$

be given by :

$$u_i(a_1,\ldots,a_n) = a_i$$

these functions are called the natural coordinate functions.

An open set in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ which is open with respect to the metric

$$d(\underline{a},\underline{b}) = \{ \sum_{i=1}^{n} (a_i - b_i)^2\}^{\frac{1}{2}}.$$

If $r \in \mathbb{Z}^+$ and if A is an open subset of $\mathbb{R}^n$ then a map

$$f : A \to \mathbb{R}$$

is called C^r on A if all partial derivatives of f up to the r^{th} are continuous. If f is merely continuous, then it is C^o on A.

When

$$A \subseteq \mathbb{R}^n$$

where A is an open subset of $\mathbb{R}^n$ then the map

$$f : A \to \mathbb{R}^m$$

is C^r on A provided that each coordinate function

$$f_i = u_i \circ f \left(i = 1,\dots,m \right)$$

is C^r on A as defined above.

The notation for coordinate functions is as follows. If $\underline{p} \in A$ then

$$f(\underline{p}) = (f_1(\underline{p}),\dots,f_m(\underline{p}))$$

so $\quad (u_i \circ f)(\underline{p}) = f_i(\underline{p})$

Henceforth, the assumption is that all functions dealt with are C^∞.

The Jacobian matrix at $\underline{z}_0$ of a C^∞ function

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m$$

is the matrix

$$J_{(f,\underline{z}_0)} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\underline{z}_0) & \frac{\partial f_1}{\partial x_2}(\underline{z}_0) & \cdots & \frac{\partial f_1}{\partial x_n}(\underline{z}_0) \\ \frac{\partial f_2}{\partial x_1}(\underline{z}_0) & \frac{\partial f_2}{\partial x_2}(\underline{z}_0) & \cdots & \frac{\partial f_2}{\partial x_n}(\underline{z}_0) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(\underline{z}_0) & \frac{\partial f_m}{\partial x_2}(\underline{z}_0) & \cdots & \frac{\partial f_m}{\partial x_n}(\underline{z}_0) \end{pmatrix}$$

This is also denoted by $f'(\underline{z}_0)$ and represents a linear transformation

from $T_{\underline{z}_0}(\mathbb{R}^n)$ to $T_{f(\underline{z}_0)}(\mathbb{R}^m)$

where

$T_{\underline{z}_0}(\mathbb{R}^n)$ = set of all vectors in $\mathbb{R}^n$ with starting point $\underline{z}_0$.

$T_{f(\underline{z}_0)}(\mathbb{R}^m)$ = set of all vectors in $\mathbb{R}^m$ with starting point at $f(\underline{z}_0)$.

The Chain Rule states that, when functions

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^\ell$$

are both C^∞ with Jacobians $f'(\underline{z})$ at $\underline{z}$ and $g'(f(\underline{z}))$ at $f(\underline{z})$

where

$f(\underline{z}) \in \text{dom } g$

then : (i) $g \circ f$ is C^∞ at $\underline{z}$

(ii) $(g \circ f)'(\underline{z}) = g'(f(\underline{z})).f'(\underline{z})$

That is, the Jacobian matrix of $g \circ f$ at $\underline{z}$ is the product of the Jacobian matrix of g at $f(z)$ with the Jacobian matrix of f at $\underline{z}$.

1.2.2 Differentiable Manifolds

If M is a set (and not necessarily a subset of $\mathbb{R}^n$) then an n-dimensional chart on M is a subset U of M together with a bijection

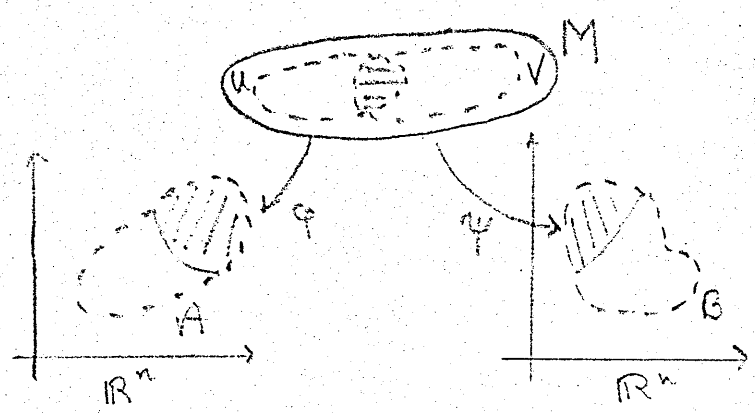
$$\phi : U \to A$$

where A is an open subset of $\mathbb{R}^n$.

Two charts (ϕ,U) and (ψ,V) are said to be C^∞ related if the maps

$$\mathbb{R}^n \xrightarrow{\phi \circ \psi^{-1}} \mathbb{R}^n \text{ and } \mathbb{R}^n \xrightarrow{\psi \circ \phi^{-1}} \mathbb{R}^n$$

are C^∞ in the Euclidean sense. So if



$$\phi : U \to A$$

and $\psi : V \to B$

then

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to A$$
$$\psi \circ \phi^{-1} : \phi(U \cap V) \to B$$

must be C^∞.

A C^∞ n-subatlas on M is a collection of C^∞-related n-charts (ϕ_h,U_h) such that

$$\bigcup_{h\in H} U_h = M$$

A maximal, C^∞ n-subatlas is called a C^∞ n-atlas. The set M becomes a C^∞ n-manifold when it is given a C^∞ n-atlas. This atlas defines the differentiable structure of M.

It is possible that set M has a differentiable structure which is specified by more than one atlas but two such atlases are equivalent if their union is again an atlas for M.

Example

Let M be the set of points which constitute the unit circle centre the origin. This set has the following two differentiable structures:

Atlas 1

is $\{(\phi,U), (\phi',U')\}$

given by:

$$U = \{\sin 2\pi s, \cos 2\pi s) : 0 < s < 1\}$$
$$\phi : U \to \mathbb{R} \text{ by } \phi(\sin 2\pi s, \cos 2\pi s) = s$$
$$U' = \{(\sin 2\pi s, \cos 2\pi s) : -\tfrac{1}{2} < s < \tfrac{1}{2}\}$$
$$\phi' : U' \to \mathbb{R} \text{ by } \phi'(\sin 2\pi s, \cos 2\pi s) = s.$$

Atlas 2

is $\{(\phi_1,U_1), (\phi_2,U_2), (\phi_3,U_3), (\phi_4,U_4)\}$

given by:

$$U_1 = \{(p_1,p_2) : p_1 > 0\} \text{ and } \phi(p_1,p_2) = p_2$$
$$U_2 = \{(p_1,p_2) : p_2 > 0\} \text{ and } \phi(p_1,p_2) = p_1$$
$$U_3 = \{(p_1,p_2) : p_1 < 0\} \text{ and } \phi(p_1,p_2) = p_2$$
$$U_4 = \{(p_1,p_2) : p_2 < 0\} \text{ and } \phi(p_1,p_2) = p_1$$

where $p_1^{\,2} + p_2^{\,2} = 1$ in each case.

To check that the union of atlases is again an atlas, it is necessary to verify that the charts U and U' overlap smoothly with charts U_1, U_2, U_3, U_4.

That is, check that maps $\phi \circ \phi_i^{-1}$, $\phi_i \circ \phi^{-1}$, $\phi' \circ \phi_i^{-1}$, $\phi_i \circ \phi'^{-1}$ are C^∞ from $\mathbb{R}$ to $\mathbb{R}$ for $i = 1,2,3,4$. For example,

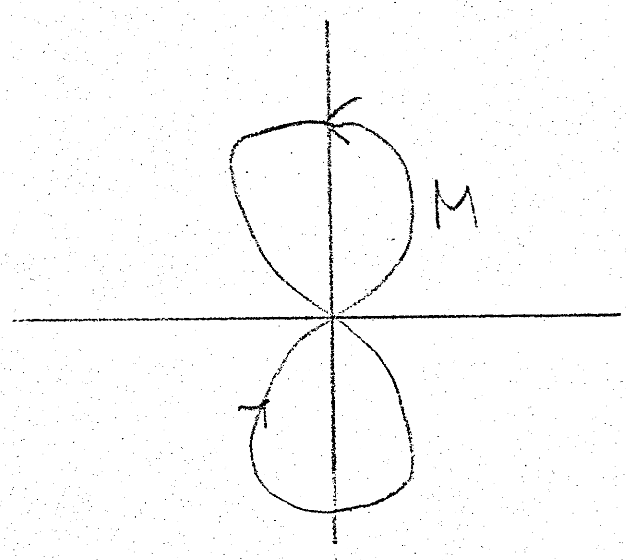
$$(\phi_3 \circ \phi'^{-1})(s) = \cos 2\pi s \ \underline{\text{is}}\ C^\infty.$$

It is possible for a manifold to possess two distinct differentiable structures as the following example shows:

<u>Example</u>

(Brickell & Clark [1])

Let M be the set $\{(\sin 2s, \sin s) \ ; \ s \in \mathbb{R}\} \subset \mathbb{R}^2$.



<u>DIFFERENTIABLE STRUCTURE 1</u>

is given by the single chart (M,ϕ) where

$$\phi : M \to \mathbb{R}$$

by the rule

$$\phi(\sin 2s, \sin s) = s \text{ for } s \in (0,2\pi).$$

DIFFERENTIABLE STRUCTURE 2

is also given by a single chart

$$\psi : M \to \mathbb{R}$$

by the rule

$$\psi(\sin 2s, \sin s) = s \text{ for } s \in (-\pi, \pi).$$

To show that these structures are distinct show that these atlases are non-equivalent. That is, show that the charts do not overlap smoothly. This follows because the map

$$\psi \circ \phi^{-1} : \mathbb{R} \to \mathbb{R}$$

given by

$$(\psi \circ \phi^{-1})(s) = \begin{cases} s & \text{if } 0 < s < \pi \\ s-\pi & \text{if } s = \pi \\ s-2\pi & \text{if } \pi < s < 2\pi \end{cases}$$

is not continuous.

1.2.3 Topology of a Manifold

A manifold is always provided with a topology by its C^∞ structure. The open sets of this topology are defined as follows.

An open set A of a manifold M is a set A such that, for any chart (ϕ,U), the set $\phi(A\cap U)$ is open as a subset of $\mathbb{R}^n$. Let $\mathcal{M}$ be the collection of all such open sets A, then:

(i) $M \in \mathcal{M}$ because if (ϕ,U) is any chart of M then

$$\phi(U\cap M) = \phi(U)$$

which is open in $\mathbb{R}^n$

$$\therefore M \in \mathcal{M}$$

(ii) $\Phi \in \mathcal{M}$ since $\phi(\Phi\cap U) = \phi(\Phi) = \Phi$ which is open in any topology.

(iii) Let $\{S_h\}_{h\in H}$ be a collection of subsets which are open in $\mathcal{M}$ then, if (ϕ,U) is a chart,

$$\phi((\bigcup_h S_h)\cap U) = \phi(\bigcup_h (S_h\cap U))$$
$$= \bigcup_h \phi(S_h\cap U)$$
$$= \text{an open subset of } \mathbb{R}^n$$

Since each $\phi(S_h\cap U)$ is open.

(iv) Take open subsets S_1, S_2 of M then, since ϕ is injective,

$$\begin{aligned}\phi(S_1 \cap S_2 \cap U) &= \phi(S_1 \cap U \cap S_2 \cap U) \\ &= \phi(S_1 \cap U) \cap \phi(S_2 \cap U) \\ &= \text{intersection of sets in } \mathbb{R}^n \\ &\Rightarrow S_1 \cap S_2 \text{ open in M.}\end{aligned}$$

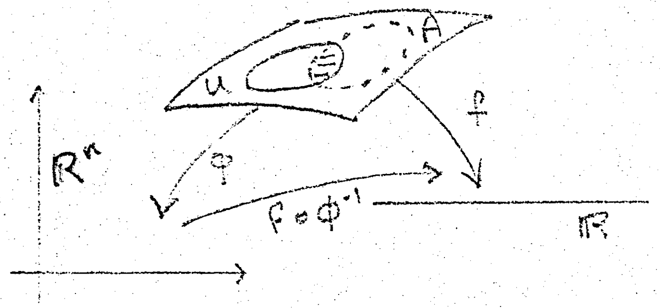
Thus the collection $\mathcal{M}$ is a topology for M called the induced topology of the manifold.

It now makes sense to speak of Hausdorff manifolds or compact manifolds and it is possible to show how the topology of a manifold affects its analysis. For example, a differentiable manifold admits a partition of unity iff it is paracompact.

1.2.4 Differentiable Functions

If A is an open set on a manifold M and if

$f : A \to \mathbb{R}$ then <u>f is C^∞</u> if $f \circ \phi^{-1}$ is C^∞ for each chart (ϕ, U) of M



where $U \cap A$ is non-empty and

$$f \circ \phi^{-1} : \phi(U \cap A) \to \mathbb{R}$$

is a function from $\mathbb{R}^n$ to $\mathbb{R}$.

In a sense, f is regarded as being a map between manifold M and manifold $\mathbb{R}$ so the natural generalization is to consider maps between a C^∞ p-manifold M and a C^∞ n-manifold N as follows:

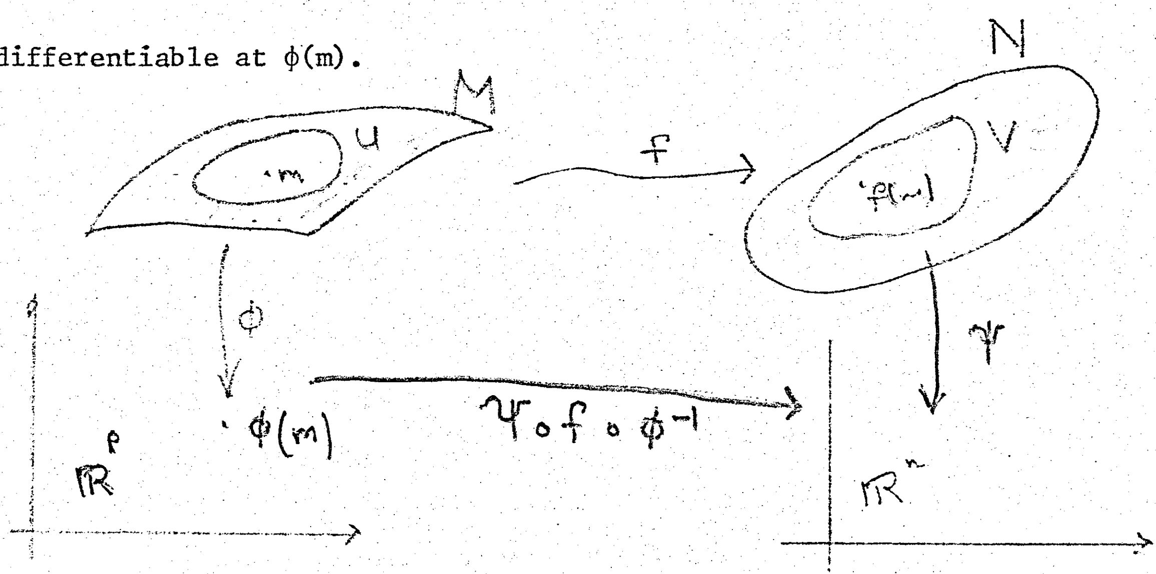
Suppose that function f has domain in M and range in N

i.e. $M \xrightarrow{f} N$

Let $m \in \text{dom } f$ and suppose that (ϕ, U) is a chart about m. If (ψ, V) is a chart of N containing f(m) then <u>f is differentiable at m</u> if

$$\mathbb{R}^p \xrightarrow{\psi \circ f \circ \phi^{-1}} \mathbb{R}^n$$

is differentiable at $\phi(m)$.



The function $\psi \circ f \circ \phi^{-1}$ is called a <u>coordinate representative of f at m</u>. This definition is independent of the two coordinate systems because if (ϕ',U') and (ψ',V') are also charts at m and f(m) respectively then, by the smooth overlap condition,

$$\psi' \circ \psi^{-1} \text{ is } C^\infty \text{ in } \mathbb{R}^n$$

$$\phi \circ \phi'^{-1} \text{ is } C^\infty \text{ in } \mathbb{R}^p$$

$$\therefore \quad (\psi' \circ \psi^{-1}) \circ (\psi \circ f \circ \phi^{-1}) \circ (\phi \circ \phi'^{-1})$$

$$= \psi' \circ f \circ \phi'^{-1}$$

is C^∞ from $\mathbb{R}^p$ to $\mathbb{R}^n$. f is <u>differentiable</u> if it is differentiable at each point in its domain. Equivalent terminology:

f is <u>C^∞</u>

f is <u>smooth</u>

The manifolds M and N are said to be <u>diffeomorphic</u> if there exists a bijection

$$f : M \to N$$

such that both f and f^{-1} are differentiable. This induces an equivalence relation on the set of all differentiable manifolds and one of the problems of differential topology is to classify manifolds up to differmorphism.

1.2.5 Tangent Vectors

Because a manifold does not necessarily reside in Euclidean space, the conventional idea of a tangent vector may have no meaning. Consequently, the definition gives a tangent vector at a point on a manifold as an operator on functions which are defined on neighbourhoods of that point. This definition generalizes the idea of directional derivative in $\mathbb{R}^n$.

Let $C^\infty(m,\mathbb{R})$ denote the set of all C^∞, real-valued functions defined on some neighbourhood of m. If $f_1, f_2 \in C^\infty(m,\mathbb{R})$ and if f_1, f_2 agree on some neighbourhood of m then f_1, f_2 are said to have the same germ at m (see Warner [7]). This induces an equivalence relation which partitions $C^\infty(m,\mathbb{R})$ into equivalence classes denoted by

$$[f_1] \text{ or } \underline{f}_1 \text{ or just } f_1.$$

$$\therefore \quad C^\infty(m,\mathbb{R})/\sim \; = \{[f];\ f \in C^\infty(m,\mathbb{R})\}$$

$$= F_m.$$

A tangent vector is defined as a derivation on F_m but a less formal definition is as follows:

Let M be a C^∞ n-manifold with $m \in M$. A tangent vector at m is a real-valued function

$$\underline{v}_m : C^\infty(m,\mathbb{R}) \longrightarrow \mathbb{R}$$

such that, whenever $f, g \in C^\infty(m,\mathbb{R})$;

1) $\underline{v}_m(f+g) = \underline{v}_m(f) + \underline{v}_m(g)$

$\underline{v}_m(cf) = c\underline{v}_m(f) \qquad (c \in \mathbb{R})$.

(2) $\underline{v}_m(f.g) = \underline{v}_m(f)g(m) + f(m)\underline{v}_m(g)$.

If we define

$$(\underline{v}_m+\underline{u}_m)f = \underline{v}_m(f) + \underline{u}_m(f)$$

and also

$$(c\underline{u}_m)f = c\{\underline{v}_m(f)\} \text{ for } c \in \mathbb{R}$$

and

$$f \in C^\infty(m,\mathbb{R})$$

then the set T_mM of all such tangent vectors $\underline{v}_m, \underline{u}_m$ is a real vector space called the <u>tangent space</u> of M at m. It is otherwise denoted by M_m.

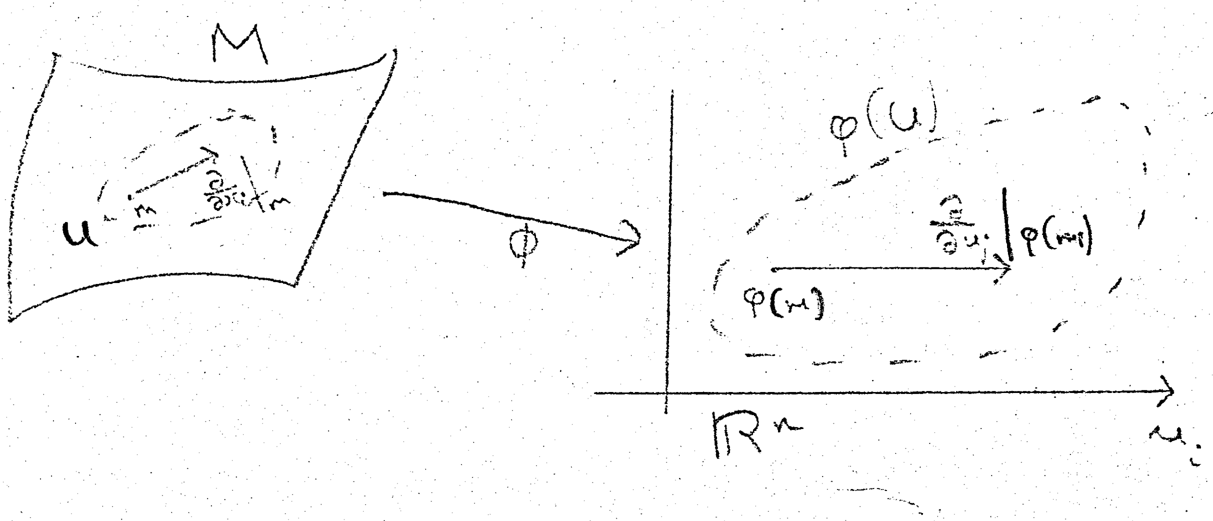
Suppose that (ϕ,U) is a chart containing m so that $x_1,x_2,\ldots,x_n$ is a <u>coordinate system</u> about m where

$$x_i = u_i \circ \phi \qquad i = 1,2,\ldots,n$$

If $f \in C^\infty(m,\mathbb{R})$ then <u>coordinate vectors</u> $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}$ are each defined by

$$\left(\frac{\partial}{\partial x_i}\right)_m (f) = \left.\frac{\partial f}{\partial x_i}\right|_m = \left.\frac{\partial(f \circ \phi^{-1})}{\partial u_i}\right|_{\phi(m)}$$

for $i = 1,2,\ldots,n$



The definition works because the function $f \circ \phi^{-1}$ goes from $\mathbb{R}^n$ to $\mathbb{R}$.

These coordinate vectors are indeed tangent vectors because they can be shown to satisfy requirements (1) and (2) on p 16. For example;

$$\left(\frac{\partial}{\partial x_i}\right)_m (f+g) = \frac{\partial}{\partial u_i}\{(f+g) \circ \phi^{-1}\}\Big|_{\phi(m)}$$

$$= \frac{\partial}{\partial u_i}\{f \circ \phi^{-1} + g \circ \phi^{-1}\}\Big|_{\phi(m)}$$

$$= \frac{\partial}{\partial u_i}(f \circ \phi^{-1})\Big|_{\phi(m)} + \frac{\partial(g \circ \phi)}{\partial}\Big|_{\phi(m)}$$

$$= \left(\frac{\partial}{\partial x_i}\right)_m (f) + \left(\frac{\partial}{\partial x_i}\right)_m (g)$$

The following theorem is proved in Hicks [4] Warner [7] and Brickell & Clark [1]:

Let M be a C^∞ n-manifold and let $x_1, \ldots, x_n$ be a coordinate system about $m \in M$. If $\underline{v}_m \in M_m$ then:

$$\underline{v}_m = \sum_{i=1}^{n} \underline{v}_m(x_i)\left(\frac{\partial}{\partial x_i}\right)_m \quad \ldots\ldots \qquad (1)$$

This means that the coordinate vectors form a basis of M_m which therefore has dimension n.

If (U,ϕ) and (V,ψ) are charts about m which respectively yield coordinate systems, $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ then

$$\frac{\partial}{\partial y_j}\bigg|_m = \sum_{i=1}^{n} \left(\frac{\partial x_i}{\partial y_j}\right)_m \frac{\partial}{\partial x_i}\bigg|_m \quad \ldots\ldots \qquad (2)$$

which is obtained by use of equation (1) above. This means that a change of basis of M_m, when affected by a change of coordinate system, is given by the matrix

$$\left(\frac{\partial x_i}{\partial y_j}\bigg|_m\right)_{n\times n}$$

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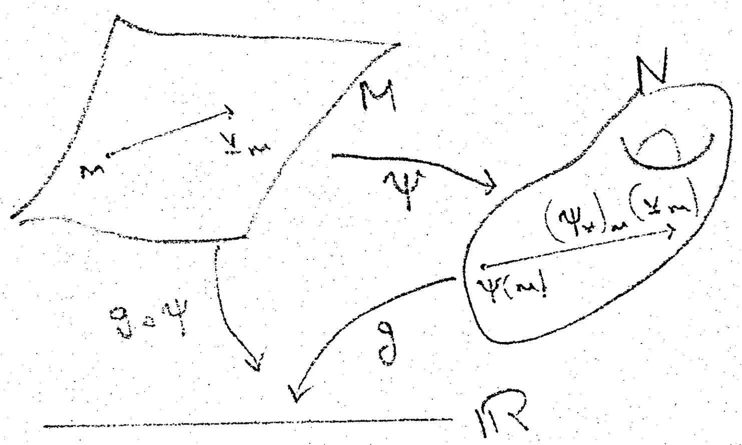
1.2.6 The Derived Mapping

If $\Psi : M \to N$ is a mapping between C^∞ manifolds and if Ψ is C^∞ then it induces a mapping

$$(\Psi_*)_m : M_m \to N_{\Psi(m)}$$

at each $m \in \operatorname{dom} \Psi$.

Thus, if $\underset{\sim}{v}_m \in M_m$ then $(\Psi_*)_m(\underset{\sim}{v}_m) \in N_{\Psi(m)}$ defined as follows:



$\forall g \in C^\infty(\Psi(m), \mathbb{R})$ define

$$\{(\Psi_*)_m(\underset{\sim}{v}_m)\}(g) = \underset{\sim}{v}_m(g \circ \Psi)$$

where

$$g \circ \Psi \in C^\infty(m, \mathbb{R})$$

so $(\Psi_*)_m \underset{\sim}{v}_m \in N_{\Psi(m)}$.

Alternative notation for $(\Psi_*)_m$ is $d\Psi_m$ and it is called the <u>derived linear map of Ψ at m</u>. It needs to be checked that $(\Psi_*)_m(\underset{\sim}{v}_m)$ satisfies the requirements (1) and (2) given in §1.2.5. That is, show

(1) $\quad (\Psi_*)_m(\underset{\sim}{v}_m)(g_1+g_2) = (\Psi_*)_m(\underset{\sim}{v}_m)(g_1) + (\Psi_*)_m(\underset{\sim}{v}_m)(g_2).$

(2) $\quad (\Psi_*)_m(\underset{\sim}{v}_m)(g_1 g_2) = (\Psi_*)_m(\underset{\sim}{v}_m)(g_1).g_2(\Psi(m)) + g_1(\Psi(m)).(\Psi_*)_m(\underset{\sim}{v}_m)(g_2).$

This verification uses the fact that $\underset{\sim}{v}_m$ itself satisfies similar properties. For example, (2) is verified as follows

$$\begin{aligned}(\Psi_*)_m(\underline{v}_m)(g_1 g_2) &= \underline{v}_m((g_1 g_2) \circ \Psi) \\ &= \underline{v}_m((g_1 \circ \Psi).(g_2 \circ \Psi)) \\ &= \underline{v}_m(g_1 \circ \Psi)(g_2 \circ \Psi)(m) + (g_1 \circ \Psi)(m).\underline{v}_m(g_2 \circ \Psi). \\ &= (\Psi_*)_m(\underline{v}_m)(g_1).g_2(\Psi(m)) + g_1(\Psi(m)).(\Psi_*)_m(\underline{v}_m)(g_2)\end{aligned}$$

By choosing a coordinate system $x_1,\ldots,x_n$ about m and another system $y_1,\ldots,y_k$ about $\psi(m)$, a Jacobian matrix can be obtained for $(\Psi_*)_m$. Let

$$X_i = \left.\frac{\partial}{\partial x_i}\right|_m \text{ and let}$$

$$Y_j = \left.\frac{\partial}{\partial y_j}\right|_{\Psi(m)} \text{ so that}$$

$$\{X_1,\ldots,X_n\}$$

is a basis of M_m and $\{Y_1,\ldots,Y_k\}$ is a basis of $N_{\Psi(m)}$

$$\therefore \quad (\Psi_*)_m\,(X_i) \in N_{\Psi(m)}$$

so the tangent vector $(\Psi_*)_m(X_i)$ can be expressed in terms of $Y_1,\ldots,Y_k$ via equation (1) on page 17:

$$\begin{aligned}(\Psi_*)_m(X_i) &= \sum_{j=1}^{k} (\Psi_*)_m(X_i)(y_j) \left.\frac{\partial}{\partial y_j}\right|_{\Psi(m)} \\ &= \sum_{j=1}^{n} X_i(y_j \circ \Psi) \left.\frac{\partial}{\partial y_j}\right|_{\Psi(m)} \\ &= \sum_{j=1}^{n} \frac{\partial(y_j \circ \Psi)}{\partial x_i} \left.\frac{\partial}{\partial y_j}\right|_{\Psi(m)}\end{aligned}$$

This means that $(\Psi_*)_m$ has matrix

$$\left(\frac{\partial(y_j \circ \Psi)}{\partial x_i}\right)_{n \times k}$$

<u>NOTE</u> The linearity of $(\Psi_*)_m$ is shown by use of the fact that

$$(\underline{v}_m + \underline{u}_m)g = \underline{v}_m(g) + \underline{u}_m(g).$$

Since $(\Psi_*)_m$ is a vector space mapping, we have:

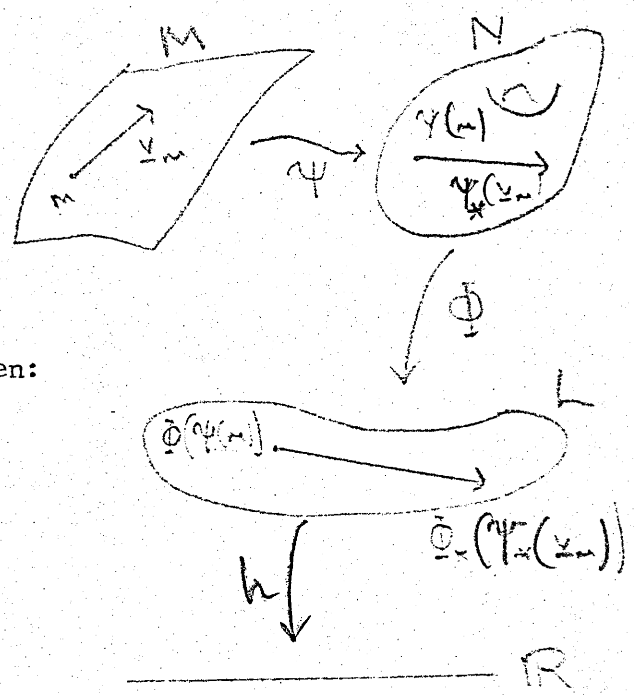
$$\dim M_m = \dim \operatorname{Ker}(\Psi_*)_m + \dim \operatorname{range}(\Psi_*)_m$$

<u>The rank of Ψ at m</u> is defined to be the rank of $(\Psi_*)_m$ - this is just the rank of the Jacobian matrix of Ψ at m.

The <u>chain rule</u> for 'Euclidean' functions may be generalized as follows:

If $\Psi : M \to N$ and $\Phi : N \to L$ are C^∞ maps between manifolds and if $m \in \operatorname{dom} \Psi$ where also $\Psi(m) \in \operatorname{dom} \Phi$ then;

$$(\Phi \circ \Psi)_{*m} = \Phi_{*\Psi(m)} \circ \Psi_{*m}$$



<u>Proof</u>

Let $h \in C^\infty(\Phi(\Psi(m)), \mathbb{R})$ and let $\underline{v}_m \in M_m$ then:

$$(\Phi \circ \Psi)_{*m}(\underline{v}_m)(h)$$

$$= \underline{v}_m(h \circ (\Phi \circ \Psi))$$

$$= \underline{v}_m((h \circ \Phi) \circ \Psi)$$

$$= \Psi_{*m}(\underline{v}_m)(h \circ \Phi)$$

$$= (\Phi_{*\Psi(m)} \circ \Psi_{*m})(\underline{v}_m)(h)$$

$$\therefore (\Phi \circ \Psi)_{*m} = \Phi_{*\Psi(m)} \circ \Psi_{*m}$$

The definition of differentiability of functions between manifolds depends strictly upon the differentiability of functions of the type

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m$$

Consequently, it is not surprising that most of the results of Euclidean calculus have analogues in the context of the theory of differentiable manifolds. One example is the above theorem and another example follows:

INVERSE FUNCTION THEOREM

Given any differentiable function $\Psi : M \to N$ with $m \in \operatorname{dom} \Psi$ then

$$\Psi_{*m} : M_m \to N_{\Psi(m)}$$

is an isomorphism iff there exists a neighbourhood $U(m)$ such that

$$\Psi | U(m)$$

is a (local) diffeomorphism.

One half of the proof utilizes the inverse function theorem at Euclidean calculus and the other half uses the chain rule above. Details are in Brickell & Clark [1].

1.2.7 Tangent Bundle

Let M be a C^∞ n-manifold and define

$$T(M) = \{(m,\underline{v}_m);\ m \in M,\ \underline{v}_m \in M_m\}$$

this can be identified with the set

$$\bigcup_{m \in M} M_m$$

A projection function $\pi : T(M) \to M$ is given by:

$$\pi(m,\underline{v}_m) = m$$

or by

$$\pi(\underline{v}_m) = m \text{ when we define } T(M) = \bigcup_{m \in M} M_m$$

If (ϕ,U) is a chart on M with $x_i = U_i \circ \phi$, define

$$\bar{x}_i = x_i \circ \pi \qquad (i = 1,\ldots,n)$$

with domain

$$\bar{\mathcal{U}} = \pi^{-1}(U) \subseteq T(M).$$

If

$$\underline{v}_m = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\bigg|_m \qquad \text{where } a_i \in \mathbb{R}$$

define $\tilde{x}_i(\underline{v}_m) = a_i$ $(i = 1,\ldots,n)$ and let $\omega(\underline{v}_m) = (a_1,\ldots,a_n)$ so $\tilde{x}_i = u_i \circ \omega$

This enables us to define a chart $(\bar{\phi}, \bar{U})$ of $T(M)$ as follows:

$$\bar{\phi} : \bar{U} \to \mathbb{R}^{2n}$$

given by:

$$\bar{\phi}(\underline{v}_m) = (\phi(m), \omega(\underline{v}_m))$$

$$= (x_1(m), \ldots, x_n(m), a_1, \ldots, a_n)$$

$$\therefore \quad \left. \begin{aligned} u_i \circ \bar{\phi} &= \bar{x}_i \\ u_{n+i} \circ \bar{\phi} &= \tilde{x}_i \end{aligned} \right\} \quad i = 1, \ldots, n.$$

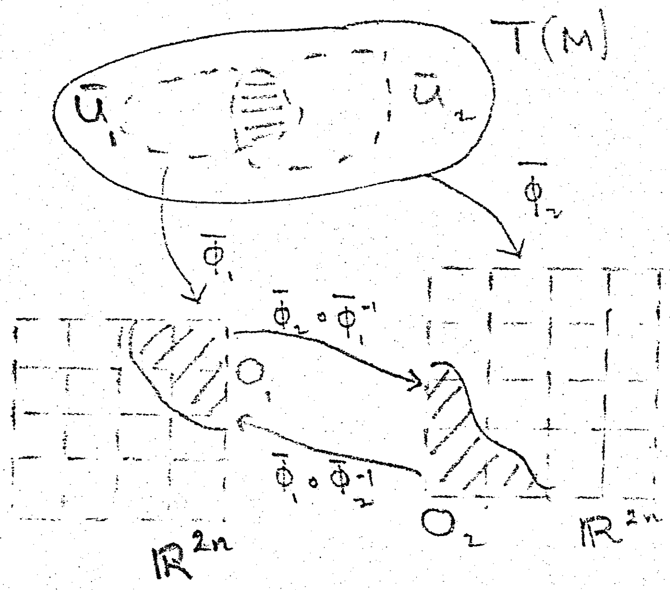
The collection of pairs $(\phi, \bar{U})$ defines a C^∞ structure on $T(M)$ which is called the <u>tangent bundle of M</u>.

$$\begin{tikzcd}
\omega(\underline{v}_m) \arrow[d, "u_{n+i}"'] & & \underline{v}_m \arrow[ll, "\omega"'] \arrow[rr, "\pi"] \arrow[dll, "\tilde{x}_i"] \arrow[drr, "\bar{x}_i = x_i \circ \pi"] & & m \arrow[d, "x_i"] \\
a_i & & & & x_i(m)
\end{tikzcd}$$

Proof

Let $(\overline{\phi}_1,\overline{U}_1)$ and $(\overline{\phi}_2,\overline{U}_2)$ both be 2n-coordinate pairs on T(M).

Show that they are C^∞-related. That is, show that



$$\overline{\phi}_2 \circ \overline{\phi}_1^{-1} : O_1 \to O_2$$

$$\overline{\phi}_1 \circ \overline{\phi}_2^{-1} : O_2 \to O_1$$

are Euclidean C^∞ when

$$O_1 \neq \emptyset \neq O_2$$

Since $\overline{\phi}_1 = (\phi_1,\omega_1)$

then

$$\overline{\phi}_1^{-1} = (\phi_1^{-1},\omega_1^{-1})$$

Hence, $\overline{\phi}_2 \circ \phi_1^{-1} = (\phi_2 \circ \phi_1^{-1}, \omega_2 \circ \omega_1^{-1})$.

Since charts (ϕ_1,U_1) and (ϕ_2,U_2) are C^∞ related then

$$\phi_2 \circ \phi_1^{-1} \text{ is } C^\infty \text{ and so is } \phi_1 \circ \phi_2^{-1}.$$

Merely show that $\omega_2 \circ \omega_1^{-1}$ is C^∞. Suppose that

$$\omega_1^{-1}(\underline{b}) = \underline{v}_m = \sum_{i=1}^{n} b_i \left(\frac{\partial}{\partial x_i}\right)_m$$

then

$$(\omega_2 \circ \omega_1^{-1})(\underline{b}) = \omega_2\left(\sum_{i=1}^{n} b_i\left(\frac{\partial}{\partial x_i}\right)_m\right)$$

where

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^{n} \left(\frac{\partial y_j}{\partial x_i}\right) \left(\frac{\partial}{\partial y_j}\right)_m$$

$$\therefore \quad (\omega_2 \circ \omega_1^{-1})(\underline{b}) = \omega_2\{\sum_{j=1}^{n} \sum_{i=1}^{n} b_i \frac{\partial y_j}{\partial x_i} (\frac{\partial}{\partial y_j})_m\}$$

$$= \sum_{i=1}^{n} b_i (\frac{\partial y_1}{\partial x_i}, \ldots, \frac{\partial y_n}{\partial x_i})_m$$

$$= \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots\cdots & \frac{\partial y_n}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial y_1}{\partial x_n} & \cdots\cdots & \frac{\partial y_n}{\partial x_n} \end{pmatrix} (\underline{b})$$

$$= J_{\phi_2 \circ \phi_1^{-1}} \Big|_{\phi(m)} (\underline{b})$$

$\therefore \omega_2 \circ \omega_1^{-1}$ is C^∞.

Similarly, $\omega_1 \circ \omega_2^{-1}$ can be shown to be C^∞. This implies that charts $(\overline{\phi}_1,\overline{U}_1)$ and $(\overline{\phi}_2,\overline{U}_2)$ overlap smoothly.

1.3.1 Vector Field on a Manifold

The idea of a vector field on a curve or surface in $\mathbb{R}^3$ can be generalized to a C^∞ n-manifold. There is a variety of definitions to choose from and they vary from intuitively acceptable definitions to more general and abstract formulations.

For example, Flanders [2] merely defines a vector field on a manifold M as being a smooth assignment of a tangent vector to each point of M. This must be taken to mean that if X is a vector field on M, then

$$X : M \to T(M)$$

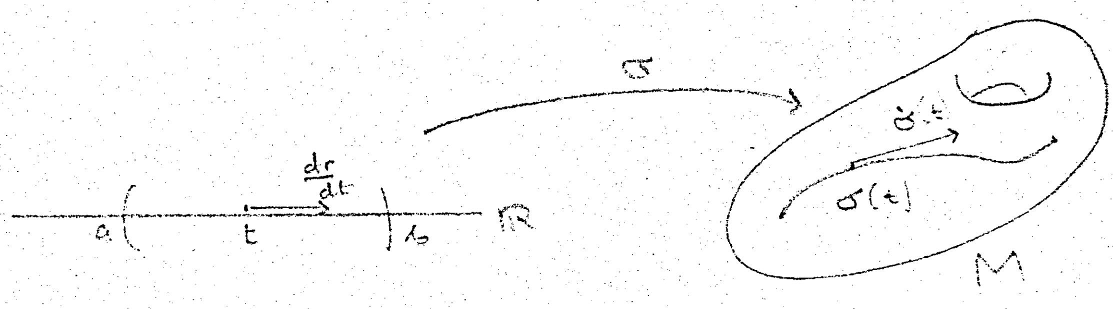
where $\quad X : m \longmapsto X_m \in M_m$

is C^∞ as a mapping between manifolds.

A more formal definition is given by Warner [7] as follows, but it depends on the idea of a curve in a manifold. A mapping

$$\sigma : (a,b) \to M$$

where (a,b) is an open interval in $\mathbb{R}$, is called a smooth curve in M if it is C^∞.



The tangent vector to the curve σ at t is the vector

$$\dot{\sigma}(t) = \sigma_{*t}\left(\frac{d}{dr}\bigg|_t\right) \in M_{\sigma(t)}$$

where the tangent space of $\mathbb{R}$ at t is spanned by the single vector

$$\frac{d}{dr}\bigg|_t$$

where $r : \mathbb{R} \to \mathbb{R}$ is the identity chart.

A mapping $\sigma : [a,b] \to M$ is also called a smooth curve in M if it extends to a C^∞ mapping of

$(a-\varepsilon, b+\varepsilon) \to M$ for some $\varepsilon \in \mathbb{R}$.

A <u>vector field X along a curve</u>,

$\sigma : [a,b] \to M$ is a mapping

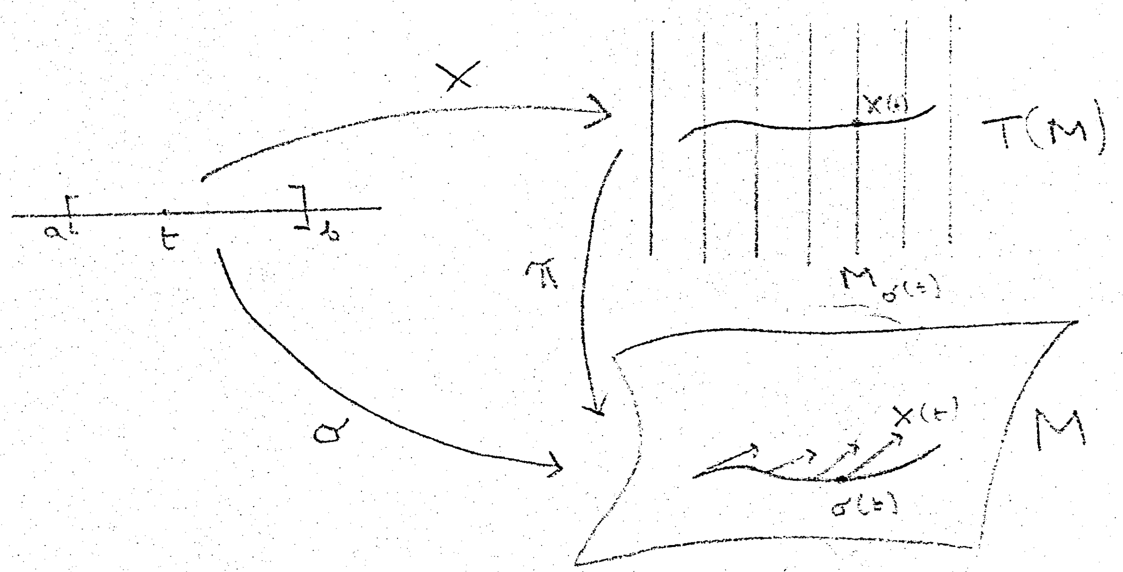
$X : [a,b] \to T(M)$

such that $\pi \circ X = \sigma$ where, as before,

$\pi : TM \to M$

is the projection

$\pi(\underline{v}_m) = m.$



This diagram suggests the terminology that X lifts σ into TM. Such a vector field is C^∞ along σ if the mapping

$$X : [a,b] \to T(M) \text{ is } C^\infty.$$

Generally, a vector field on an open set U in M is a lifting of U into T(M). Thus, X is a vector field on U if

$$X : U \to TM$$

such that

$$\pi \circ X : U \to U$$

is the identity map.

The vector field X is smooth or C^∞ if it is C^∞ as a mapping.

Denote the set of all C^∞ vector fields on U by χ_u and define binary operations as follows:

Let $X, Y \in \chi_u$, $a, b \in \mathbb{R}$ and let $f, g \in C^\infty(U, \mathbb{R})$ then:

(i) $X+Y$ is given by $(X+Y)_m = X_m + Y_m$

(ii) $(a+b)X$ is given by

$$\{(a+b)X\}_m = aX_m + bX_m$$
$$= (a+b)X_m$$

(iii) fX is given by

$$(fX)_m = f(m)X_m$$

(iv) $(f+g)X$ is given by

$$\{(f+g)X\}_m = (f+g)(m)X_m.$$

With these operations, χ_u becomes a real vector space and a module over $C^\infty(U, \mathbb{R})$. For notation, use

$$X(m) = X_m \text{ when } m \in U \text{ and } X : U \to TM.$$

The definition of a vector field X as being C^∞ has several equivalent formulations as follows:

(a) $X : U \to T(M)$ is a C^∞ mapping.

(b) If (V, Ψ) is a chart on M and if

$a_i : V \to \mathbb{R}$ are defined by

$$X|U \cap V = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$$

then the functions a_i are C^∞ on $U \cap V$.

(c) If W is open in M and if $f \in C^{\infty}(W, \mathbb{R})$ then

$$X(f) \in C^{\infty}(W, \mathbb{R})$$

The proof that statements (a), (b) and (c) are equivalent can be found in Warner [7].

1.3.2 Invariant Vector Fields

When $\Phi : M \to M$ is a differmorphism and when X is a vector field on M which is such that

$$\Phi_* \circ X = X \circ \Phi$$

then X is said to be invariant under Φ.

In classical vector field theory, such a field is sometimes called a pseudo vector field. The following theorems provides a criterion for discerning such fields on $\mathbb{R}^n$.

Theorem

Let X be a vector field on $\mathbb{R}^n$ having coordinate functions $f_1,\ldots,f_n$. In terms of the identity chart

$$\phi = (x_1,\ldots,x_n) \text{ on } \mathbb{R}^n$$

we have

$$X = \sum_{i=1}^{n} f_i(x_1,\ldots,x_n)\frac{\partial}{\partial x_i}$$

Then, X is invariant under $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ iff

$$f_j \circ \Phi = \sum_{i=1}^{n} f_i \frac{\partial \Phi_j}{\partial x_i} \qquad (j = 1,\ldots,n)$$

where $\Phi = (\phi_1,\ldots,\phi_n)$

For a proof of this see Brickell & Clark , Chapter 7. [1]

Example

Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be the inversion

$$\Phi \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 \\ -x_3 \end{pmatrix}$$

$$\therefore \qquad \phi_j \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -x_j \qquad\qquad (j = 1,2,3)$$

Let $\underline{H} = (H_1, H_2, H_3)$ be the magnetic field where

$$H_i : \mathbb{R}^3 \to \mathbb{R}$$

(for $i = 1,2,3$).

Regard these coordinate functions as being C^∞. The above theorem gives:

$\underline{H}$ is invariant under Φ <u>iff</u>

$$H_j \circ \Phi = \sum_{i=1}^{3} H_i \frac{\partial \phi_j}{\partial x_i}$$

where $\dfrac{\partial \Phi_j}{\partial x_i} = \delta^{i,j}$

$\therefore$ $\underline{H}$ is invariant <u>iff</u>

$$H_j \circ \phi = -H_j$$

<u>iff</u> $H_j(-x_1, -x_2, -x_3) = -H_j(x_1, x_2, x_3)$

The invariance of vector fields is a property of the coordinate functions of those fields.

1.4.1 Summary

This chapter has outlined some of the basic concepts and techniques which are required for the subsequent development of this dissertation. Firstly, the idea of a differentiable manifold as being a space which is locally like Euclidean space.

This has been defined so that the basic concepts of Euclidean calculus can be carried over.

From the idea of directional derivative in $\mathbb{R}^3$, the notion of tangent vector to a manifold was obtained. Mappings between manifolds were introduced and the corresponding derived mapping was defined. In particular, the existence of a diffeomorphism between manifolds means that they have the same sort of differentiable structure. In this way, manifolds may be classified up to diffeomorphism.

Finally, vector fields on an open subset of a manifold M are analogous to those on curves and surfaces in $\mathbb{R}^3$.

Chapter 2

EXTERIOR ALGEBRA, DIFFERENTIAL FORMS AND EXTERIOR DERIVATIVE

2.1.1 Tensors and Tensor Product on a Vector Space

Let V be a real vector space of dimension n. Denote the p-fold cartesian product of V with itself by V^p. That is;

$$V^p = V \times V \times \ldots \times V$$

Any real-valued, multilinear function on V^p, say

$$\alpha : V^p \to \mathbb{R}$$

is called a <u>p-co tensor on V</u>.

The set $T^p(V)$ of all p-co tensors on V becomes a vector space when we define

$$(\alpha+\beta)(\underline{v}_1,\ldots,\underline{v}_p) = \alpha(\underline{v}_1,\ldots,\underline{v}_p) + \beta(\underline{v}_1,\ldots,\underline{v}_p)$$

$$(a\alpha)(\underline{v}_1,\ldots,\underline{v}_p) = a.\alpha(\underline{v}_1,\ldots,\underline{v}_p)$$

where $\alpha,\beta \in T^p(V)$ and $a \in \mathbb{R}$.

It is possible to combine members of $T^p(V)$ and $T^q(V)$ by the idea of <u>tensor product</u>. Thus, if

$$\alpha \in T^p(V) \text{ and } \beta \in T^q(V)$$

then define

$$\alpha \otimes \beta \in T^{p+q}(V)$$

by the rule:

$$(\alpha \otimes \beta)(\underline{v}_1,\ldots,\underline{v}_p,\underline{v}_{p+1},\ldots,\underline{v}_{p+q})$$

$$= \alpha(\underline{v}_1,\ldots,\underline{v}_p).\beta(\underline{v}_{p+1},\ldots,\underline{v}_{p+q})$$

When $p \neq q$ notice that $\alpha \otimes \beta \neq \beta \otimes \alpha$ so tensor product is not a commutative binary operation. However, the following properties are easily checked:

(a) $(\alpha_1+\alpha_2) \otimes \beta = \alpha_1 \otimes \beta + \alpha_2 \otimes \beta.$

(b) $\alpha \otimes (\beta_1+\beta_2) = \alpha \otimes \beta_1 + \alpha \otimes \beta_2.$

(c) $(a\alpha) \otimes \beta = \alpha \otimes (a\beta) = a(\alpha \otimes \beta)$

(d) $(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma)$

where $\alpha, \alpha_1, \alpha_2 \in T^p(V)$

$\beta, \beta_1, \beta_2 \in T^q(V)$

$a \in \mathbb{R}$ and $\gamma \in T^r(V)$

The first of these will be proved to illustrate the procedure.

Proof (a)

As stated above, let α_1, α_2 be p-co tensors and let β be a q-co tensor. If

$$(\underline{v}_1, \ldots, \underline{v}_p, \underline{v}_{p+1}, \ldots, \underline{v}_{p+q}) \in V^{p+q}$$

then:

$$[(\alpha_1+\alpha_2) \otimes \beta]\,(\underline{v}_1, \ldots, \underline{v}_p, \underline{v}_{p+1}, \ldots, \underline{v}_{p+q})$$

$$= (\alpha_1+\alpha_2)(\underline{v}_1, \ldots, \underline{v}_p).\beta(\underline{v}_{p+1}, \ldots, \underline{v}_{p+q})$$

$$= [\alpha_1(\underline{v}_1, \ldots, \underline{v}_p) + \alpha_2(\underline{v}_1, \ldots, \underline{v}_p)].\beta(\underline{v}_{p+1}, \ldots, \underline{v}_{p+q})$$

$$= \alpha_1(\underline{v}_1, \ldots, \underline{v}_p).\beta(\underline{v}_{p+1}, \ldots, \underline{v}_{p+q})$$

$$+ \alpha_2(\underline{v}_1, \ldots, \underline{v}_p).\beta(\underline{v}_{p+1}, \ldots, \underline{v}_{p+q})$$

$$= (\alpha_1 \otimes \beta)(\underline{v}_1,\dots,\underline{v}_p,\underline{v}_{p+1},\dots,\underline{v}_{p+q})$$

$$+ (\alpha_2 \otimes \beta)(\underline{v}_1,\dots,\underline{v}_p,\underline{v}_{p+1},\dots,\underline{v}_{p+q})$$

$$= [\alpha_1 \otimes \beta + \alpha_2 \otimes \beta]\ (\underline{v}_1,\dots,\underline{v}_p,\underline{v}_{p+1},\dots,\underline{v}_{p+q})$$

Hence,

$$(\alpha_1+\alpha_2) \otimes \beta = \alpha_1 \otimes \beta + \alpha_2 \otimes \beta. \qquad //$$

Since $T'(V)$ is just the dual space of V and since, for any $p \in \mathbb{Z}^+$, we have

$$\alpha_1,\alpha_2,\dots,\alpha_p \in T'(V)$$

$$\Rightarrow \alpha_1 \otimes \dots \otimes \alpha_p \in T^p(V)$$

we can describe $T^p(V)$ in terms of $T'(V)$ as follows:

Theorem

When vector space V has a basis $\{\underline{e}_1,\dots,\underline{e}_n\}$ and dual basis $\{w_1,\dots,w_n\}$ i.e. $w_i(\underline{e}_j) = \delta_{ij}$ then a basis of $T^p(V)$ consists of all products of the type $w_{i_1} \otimes \dots \otimes w_{i_p}$ where $1 \le i_1,i_2,\dots,i_p \le n$ so $T^p(V)$ has dimension n^p.

Proof

By definition of tensor product,

$$(w_{i_1} \otimes \dots \otimes w_{i_p})(\underline{e}_{j_1},\dots,\underline{e}_{j_p})$$

$$= w_{i_1}(\underline{e}_{j_1}) \cdot \dots \cdot w_{i_p}(\underline{e}_{j_p})$$

$$= \delta_{i_1 j_1} \cdot \dots \cdot \delta_{i_p j_p}$$

$$= \begin{cases} 1 \text{ if } i_1 = j_1,\ \dots,\ i_p = j_p \\ 0 \text{ otherwise} \end{cases}$$

Consider the members $\underline{v}_1,\dots,\underline{v}_p$ of V with

$$\underline{v}_i = \sum_{j=1}^{n} a_{ij}\underline{e}_j \qquad \text{for } a_{ij} \in \mathbb{R}$$

If $\alpha \in T^p(V)$ then:

$$\alpha(\underline{v}_1,\dots,\underline{v}_p) = \alpha\left(\sum_{j_1=1}^{n} a_{ij_1}\underline{e}_{j_1},\underline{v}_2,\dots,\underline{v}_p\right) \quad \text{where} \quad \underline{v}_1 = \sum_{j=1}^{n} a_{1j}\,\underline{e}_j$$

$$= \sum_{j_1=1}^{n} a_{ij_1}\alpha(\underline{e}_{j_1},\underline{v}_2,\dots,\underline{v}_p)$$

$$= \sum_{j_1=1}^{n}\ \sum_{j_2=1}^{n} a_{ij_1}a_{2j_2}\alpha(\underline{e}_{j_1},\underline{e}_{j_2},\dots,\underline{v}_p)$$

$$\vdots$$

$$= \sum_{j_1=1}^{n}\ \sum_{j_2=1}^{n} \dots \sum_{j_p=1}^{n} a_{ij},\dots,a_{pj_p}\alpha(\underline{e}_{j_1},\dots,\underline{e}_{j_p})$$

using the fact that α is p-linear. Hence

$$\alpha(\underline{v}_1,\dots,\underline{v}_p) = \sum_{j_1,\dots,j_p=1}^{n} a_{ij_1},\dots,a_{pj_p}\alpha(\underline{e}_{j_1},\dots,\underline{e}_{j_p})$$

However,

$$\underline{v}_i = \sum_{j=1}^{n} a_{ij}\underline{e}_j \Rightarrow w_j(\underline{v}_i) = a_{ij}$$

so $$a_{1j_1}, \ldots, a_{pj_p} = w_{j_1}(\underline{v}_1) \cdot \ldots \cdot w_{j_p}(\underline{v}_p)$$

$$= (w_{j_1} \otimes \ldots \otimes w_{j_p})(\underline{v}_1, \ldots, \underline{v}_p)$$

which gives:

$$\alpha(\underline{v}_1, \ldots, \underline{v}_p)$$

$$= \sum_{j_1,\ldots,j_p=1}^{n} \alpha(\underline{e}_{j_1}, \ldots, \underline{e}_{j_p})(w_{j_1} \otimes \ldots \otimes w_{j_p})(\underline{v}_1, \ldots, \underline{v}_p)$$

$$\therefore \quad \alpha = \sum_{j_1,\ldots,j_p=1}^{n} \alpha(\underline{e}_{j_1}, \ldots, \underline{e}_{j_p})(w_{j_1} \otimes \ldots \otimes w_{j_p})$$

This proves that the products $w_{j_1} \otimes \ldots \otimes w_{j_p}$ span $T^p(V)$.

To show that the set of all such objects are linearly independent, assume that there are real numbers $a_{i_1,\ldots,i_p}$ such that

$$\sum_{i_1,\ldots,i_p=1}^{n} a_{i_1,\ldots,i_p} w_{i_1} \otimes \ldots \otimes w_{i_p} = 0$$

Applying both sides of this equation to

$$(\underline{e}_{i_1}, \ldots, \underline{e}_{i_p}) \in T^p(V) \text{ gives } a_{i_1,\ldots,i_p} = 0$$

This proves linear independence and completes the proof. //

The above proof is an amplification of that given by Spivak [6].

Certain p-co tensors called p-forms can be selected and their distinguishing feature is simply that they are alternating. That is, $\alpha \in T^p(V)$ is a p-form on V if

$$\alpha \circ \pi = (\operatorname{sgn} \pi)\, \alpha$$

where π is a permutation on the set $\{1,2,\ldots,p\}$ and

$$(\alpha \circ \pi)(\underline{v}_1,\ldots,\underline{v}_p) = \alpha(\underline{v}_{\pi(1)},\ldots,\underline{v}_{\pi(p)})$$

As usual, $\operatorname{sgn} \pi = \pm 1$.

The set of all p-forms on V is denoted by $F^p(V)$. In the particular case of $p = 1$ we have

$$T'(V) = F'(V)$$

meaning that any 1-co tensor is a 1-form and vice versa. To complete the definition define

$$T^0(V) = F^0(V) = \mathbb{R}$$

The reference for this section is Hicks [4] and Spivak [6].

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2.1.2 Exterior Algebra over a Vector Space

Let α and β be p and q-forms over a vector space V. Define their exterior product $\alpha \wedge \beta$ by :

$$\alpha\wedge\beta = \frac{1}{p!q!} \Sigma \quad (\operatorname{sgn} \pi)(\alpha \otimes \beta) \circ \pi$$

where the sum is taken over all permutations of the set $\{1,2,\ldots,(p+q)\}$ so if $\alpha \in F^p(V)$ and if $\beta \in F^q(V)$ then

$$\alpha \wedge \beta \in F^{p+q}(V).$$

An alternative, equivalent formulation is to give:

$$\alpha\wedge\beta = \Sigma \ (\operatorname{sgn} \pi)(\alpha \otimes \beta) \circ \pi \qquad \ldots(1)$$

where the sum is now over all shuffle permutations for p and q.

Properties

(a) $\alpha\wedge\beta = (-1)^{pq}\ \beta\wedge\alpha$

(b) $\alpha\wedge(\beta_1+\beta_2) = \alpha\wedge\beta_1 + \alpha\wedge\beta_2$

where $\alpha \in F^p(V)$ and $\beta_1,\beta_2 \in F^q(V)$

(c) $(\alpha\wedge\beta)\wedge\gamma = \alpha\wedge(\beta\wedge\gamma)$.

Proof

(a) Using equation (1),

$$(\alpha\wedge\beta)(\underline{v}_1,\ldots,\underline{v}_p,\underline{v}_{p+1},\ldots,\underline{v}_{p+q})$$

$$= \sum_{\pi} (\operatorname{sgn} \pi)\alpha(\underline{v}_{\pi(1)},\ldots,\underline{v}_{\pi(p)})\beta(\underline{v}_{\pi(p+1)},\ldots,\underline{v}_{\pi(p+q)})$$

Specifically, π is a permutation on $\{1,2,\ldots,p+q)$ satisfying

$$\pi(1) < \ldots < \pi(p) \text{ and } \pi(p+1) < \ldots < \pi(p+q).$$

Suppose that τ is the permutation taking $\{1,\ldots,p+q\}$ into the sequence

$$\{p+1,\ldots,p+q,1,\ldots,p\}$$

$$\therefore \quad 1 \leq i \leq p \Longrightarrow \pi(i) = (\pi \circ \tau)(q+i)$$

and $\quad p+1 \leq j \leq p+q \Rightarrow \pi(j) = (\pi \circ \tau)(j-p)$

If $\sigma = \pi \circ \tau$ then the inequalities above imply that σ satisfies

$$\sigma(1) < \ldots < \sigma(q) \text{ and } \sigma(q+1) < \ldots < \sigma(q+p)$$

that is, σ is also a shuffle permutation.

Now use the following results:

(a) $\quad \operatorname{sgn} \pi = (\operatorname{sgn} \sigma)(\operatorname{sgn} \tau)$

which follows since $\pi = \sigma \circ \tau^{-1}$ and $\operatorname{sgn} \tau = \operatorname{sgn} \tau^{-1}$

(b) $\quad \operatorname{sgn} \tau = (-1)^{pq}$

since mapping the sequence $\{1,\ldots,p,p+1,\ldots,p+q\}$ to the sequence $\{p+1,\ldots,p+q,1,\ldots,p\}$

is effected by pq transpositions each of sign -1.

Hence;

$$(\alpha\wedge\beta)(\underline{v}_1,\ldots,\underline{v}_{p+q})$$

$$= \sum_{\pi} (\operatorname{sgn} \pi)\alpha(\underline{v}_{\pi(1)},\ldots,\underline{v}_{\pi(p)}).\beta(\underline{v}_{\pi(p+1)},\ldots,\underline{v}_{\pi(p+q)})$$

$$= (-1)^{pq} \sum_{\sigma} (\operatorname{sgn} \sigma)\alpha(\underline{v}_{\sigma(q+1)},\ldots,\underline{v}_{\sigma(q+p)}).\beta(\underline{v}_{\sigma(1)},\ldots,\underline{v}_{\sigma(q)})$$

$$= (-1)^{pq} \sum_{\sigma} (\operatorname{sgn} \sigma)\beta(\underline{v}_{\sigma(1)},\ldots,\underline{v}_{\sigma(q)}).\alpha(\underline{v}_{\sigma(q+1)},\ldots,\underline{v}_{\sigma(q+p)})$$

$$= (-1)^{pq}(\beta\wedge\alpha)(\underline{v}_1,\ldots,\underline{v}_{p+q})$$

This means that

$$\alpha\wedge\beta = (-1)^{pq}(\beta\wedge\alpha).$$

(b) This is easily proved by using the corresponding result

$$\alpha \otimes (\beta_1+\beta_2) = \alpha \otimes \beta_1 + \alpha \otimes \beta_2$$

from the previous section.

(c) Proved in Goetz [3], p.333.

When V is a real, n-dimensional vector space, define

$$F(V) = \sum_{p\geq 0} F^p(V)$$

where the summation is 'weak direct' (Hicks [4], p51). This means that any member of F(V) consists of a finite sum of the type:

$$\alpha_1 + \alpha_2 + \ldots + \alpha_r \text{ where } \alpha_1 \in F^{p_1}(V),\ldots,\alpha_r \in F^{p_r}(V)$$

By virtue of the distributivity of exterior product over addition, F(V) is an algebra over $\mathbb{R}$ called the <u>exterior algebra over V</u>.

The following theorem is strictly analogous to that given in Section 2.1.1 for tensors. It enables one to construct a basis for $F^p(V)$ out of a basis for the dual space $F'(V)$ of V when $p > 1$.

Theorem

If $\{\underline{e}_1,\ldots,\underline{e}_n\}$ is a basis of V and if $\{w_1,\ldots,w_n\}$ is the dual basis then

$$\{w_{i_1} \wedge w_{i_2} \wedge \ldots \wedge w_{i_p} : 1 \leq i_1 < i_2 < \ldots < i_p \leq n\}$$

is a basis of $F^p(V)$. Further

$$\dim F^p(V) = \binom{n}{p}.$$

Proof

Since $F^p(V)$ is a subspace of $T^p(V)$ then the theorem of p 39 may be applied to any p-form α on V. That is,

$$\alpha \in F^p(V) \subseteq T^p(V)$$

$$\Rightarrow \alpha = \sum_{i_1,\ldots,i_p=1}^{n} a_{i_1 \ldots i_p} w_{i_1} \otimes \ldots \otimes w_{i_p} \qquad \ldots (1)$$

where $\{w_1,\ldots,w_n\}$ is the dual basis of V.

The following result is needed. Any $\alpha \in F^p(V)$ can be expressed as;

$$\alpha = \frac{1}{p!} \sum_{\pi} (\operatorname{sgn} \pi)\, \alpha \circ \pi$$

which is proved as follows. Since any permutation is a composition of transpositions, it can be shown that

$$\alpha \circ \pi = (\operatorname{sgn} \pi)\alpha$$

for all α. Hence,

$$\frac{1}{p!} \sum_{\pi} (\operatorname{sgn} \pi)\alpha(\underline{v}_{\pi(1)},\ldots,\underline{v}_{\pi(p)})$$

$$= \frac{1}{p!} \sum_{\pi} (\operatorname{sgn} \pi)(\alpha \circ \pi)(\underline{v}_1,\ldots,\underline{v}_p)$$

$$= \frac{1}{p!} \sum_{\pi} (\operatorname{sgn} \pi)(\operatorname{sgn} \pi)\alpha(\underline{v}_1,\ldots,\underline{v}_p)$$

$$= \frac{1}{p!} \sum_{\pi} \alpha(\underline{v}_1,\ldots,\underline{v}_p) = \frac{1}{p!} \cdot p!\ \alpha(\underline{v}_1,\ldots,\underline{v}_p)$$

because $\sum_{\pi}$ has $p!$ terms.

$$\therefore \quad \frac{1}{p!} \sum_{\pi} (\operatorname{sgn} \pi)\alpha \circ \pi = \alpha \qquad \ldots\ldots.(2)$$

Applying this to α as specified in equation (1) above, gives:

$$\alpha = \frac{1}{p!} \sum_{\pi} (\operatorname{sgn} \pi)\left(\sum_{i_1,\ldots,i_p=1}^{n} a_{i_1\ldots i_p}\, w_{i_1} \otimes \ldots \otimes w_{i_p}\right) \circ \pi$$

$$= \frac{1}{p!} \sum_{i_1,\ldots,i_p=1}^{n} a_{i_1,\ldots,i_p} \sum_{\pi} (\operatorname{sgn} \pi)(w_{i_1} \otimes \ldots \otimes w_{i_p}) \circ \pi$$

But

$$w_{i_1} \wedge \ldots \wedge w_{i_p} = \frac{1}{1!\ldots1!} \sum_{\pi} (\operatorname{sgn} \pi)(w_{i_1} \otimes \ldots \otimes w_{i_p}) \circ \pi$$

$$= \sum_{\pi} (\operatorname{sgn} \pi)(w_{i_1} \otimes \ldots \otimes w_{i_p}) \circ \pi$$

$$\therefore \quad \alpha = \frac{1}{p!} \sum_{i_1,\ldots,i_p=1}^{n} a_{i_1,\ldots,i_p}\, w_{i_1} \wedge \ldots \wedge w_{i_p}$$

This proves that the p-forms $w_{i_1} \wedge \ldots \wedge w_{i_p}$ span $F^p(V)$.

Linear independence is shown as for tensors in the previous section. //

Using property (a) from page 43 where $\alpha = w_i$ and $\beta = w_j$ are 1-forms, we see that $w_i \wedge w_j = -w_j \wedge w_i$

$\therefore \quad w_i \wedge w_i = 0$ for any 1-form.

Therefore, in selecting 1-forms from the basis $\{w_1,\ldots,w_n\}$ of $F'(V)$ to construct terms

$$w_{i_1} \wedge \ldots \wedge w_{i_p}$$

no repetition of choice is allowed. The number of such basis elements is $\binom{n}{p}$

$\therefore \quad \dim F^p(V) = \binom{n}{p}$.

The above theorem implies that $F(V)$ has a basis which is the union of all sets of the form:

$$\{w_{i_1} \wedge \ldots \wedge w_{i_p} ; 1 \le i_1 < \ldots < i_p \le n\}$$

for $1 \le p \le n$, together with $1 \in F^0(V)$. This basis therefore corresponds to the set of all subsets of the set $\{w_1,\ldots,w_n\}$

$\therefore \quad \dim F(V) = 2^n$

Finally, the property (a) noted previously

i.e. $\alpha \wedge \beta = (-1)^{pq}\, \beta \wedge \alpha$

gives $\qquad \alpha \wedge \beta = -\beta \wedge \alpha$ when p,q odd.

$\qquad \alpha \wedge \alpha = 0 \qquad$ for any odd form.

2.1.3 Induced Maps on Tensors and Forms

If $f : V \to W$ is a vector space linear transformation, is there any correspondence between

$T^p(V)$ and $T^p(W)$?

$F^p(V)$ and $F^p(W)$?

Using the notation of Hicks [4], define a map

$$f^* : T^p(W) \to T^p(V)$$

as follows.

If $\alpha \in T^p(W)$ then $f^*(\alpha) \in T^p(V)$ is defined by

$$f^*(\alpha)(\underline{v}_1,\ldots,\underline{v}_p) = \alpha(f(\underline{v}_1),\ldots,f(\underline{v}_p))$$

Listed below are the main properties.

(a) $f^*(\alpha_1+\alpha_2) = f^*(\alpha_1) + f^*(\alpha_2)$

Proof

$$f^*(\alpha_1+\alpha_2)(\underline{v}_1,\ldots,\underline{v}_p)$$

$$= (\alpha_1+\alpha_2)(f(\underline{v}_1),\ldots,f(\underline{v}_p))$$

$$= \alpha_1(f(\underline{v}_1),\ldots,f(\underline{v}_p)) + \alpha_2(f(\underline{v}_1),\ldots,f(\underline{v}_p))$$

$$= f^*(\alpha_1)(\underline{v}_1,\ldots,\underline{v}_p) + f^*(\alpha_2)(\underline{v}_1,\ldots,\underline{v}_p)$$

$$= [f^*(\alpha_1) + f^*(\alpha_2)]\,(\underline{v}_1,\ldots,\underline{v}_p)$$

which proves the result.

(b) $f^*(\alpha_1 \otimes \alpha_2) = f^*(\alpha_1) \otimes f^*(\alpha_2)$.

Proof

Take $\alpha_1 \in T^p(W)$ and $\alpha_2 \in T^q(W)$ and consider

$$f^*(\alpha_1 \otimes \alpha_2)(\underline{v}_1,\ldots,\underline{v}_p) = (\alpha_1 \otimes \alpha_2)(f(\underline{v}_1),\ldots,f(\underline{v}_p))$$

$= \alpha_1(f(\underline{v}_1),\ldots,f(\underline{v}_p)).\alpha_2(f(\underline{v}_1),\ldots,f(\underline{v}_p))$

$= f^*(\alpha_1)(\underline{v}_1,\ldots,\underline{v}_p).f^*(\alpha_2)(\underline{v}_1,\ldots,\underline{v}_p)$

$= [f^*(\alpha_1) \otimes f^*(\alpha_2)]\ (\underline{v}_1,\ldots,\underline{v}_p)$

$\therefore \quad f^*(\alpha_1 \otimes \alpha_2) = f^*(\alpha_1) \otimes f^*(\alpha_2)$.

(c) $f^*(\alpha_1 \wedge \alpha_2) = f^*(\alpha_1) \wedge f^*(\alpha_2)$

Proof

Straightforward by use of (b).

2.1.4 Inner Product on an Exterior Algebra.

When V is a vector space with an inner product $<\ ,\ >$ then there is a natural isomorphism of V with $F'(V)$. Suppose that, under this isomorphism, any $\underline{v} \in V$ is mapped to $v \in F'(V)$. We now get an inner product $<\ ,\ >$ on $F'(V)$ given by:

$$<v_1, v_2> = <\underline{v}_1, \underline{v}_2> \text{ for } v_1, v_2 \in F'(V)$$

An inner product $\leq\ \ \geq$ can be defined on $F^p(V)$ and on $F(V)$ as follows. Take

$$\alpha = w_1 \wedge \dots \wedge w_p \in F^p(V)$$
$$\beta = v_1 \wedge \dots \wedge v_q \in F^q(V)$$

where $w_i, v_j \in F'(V)$. Now define

$$\begin{aligned} \leq\alpha, \beta\geq &= \det(<w_i, v_j>) && \text{if } p = q \\ &= 0 && \text{if } p \neq q \end{aligned}$$

Since $\lambda\alpha = (\lambda w_1) \wedge \dots \wedge w_p$, then

$$\leq\lambda\alpha, \beta\geq = \lambda\leq\alpha, \beta\geq$$

Also, since

$$\leq w_1 \wedge \dots \wedge (w_i + w_i') \wedge \dots \wedge w_p, v_1 \wedge \dots \wedge v_p\geq$$

$$= \leq w_1 \wedge \dots \wedge w_i \wedge \dots \wedge w_p, v_1 \wedge \dots \wedge v_p\geq$$

$$+ \leq w_1 \wedge \dots \wedge w_i' \wedge \dots \wedge w_p, v_1 \wedge \dots \wedge v_p\geq$$

then extend $\leq\ ,\ \geq$ to the whole of $F^p(V)$ by defining

$$\leq\alpha + \alpha', \beta\geq = \leq\alpha, \beta\geq + \leq\alpha', \beta\geq$$

The symmetry of $\leq\ ,\ \geq$ follows from the fact that $<\ ,\ >$ is itself symmetric and the fact that the determinant is also symmetric

$$\underline{\text{i.e.}}\ \det M = \det M^t$$

Finally, $\leq\ \ \geq$ is non-degenerate. To show this, let $e_1,\ldots,e_n$ be an orthonormal basis of $F(V)$. Then the p-forms

$$e^H = e_{h_1} \wedge \ldots \wedge e_{h_p}$$

where $H = \{h_1 < \ldots < h_p\}$ form a basis of $F^p(V)$.

Now

$$\leq e^H, e^K \geq\ = \det(<e_{h_i}, e_{k_i}>)$$

If $H \neq K$, then there is a zero row in this determinant

$$\therefore \quad \leq e^H, e^K \geq\ = 0$$

If $H = K$, then

$$\det(<e_{h_i}, e_{k_j}>) = \det(\pm I_n)$$

Hence, $\leq e^H, e^K \geq\ = \pm\delta^{H,K}$ which means that the inner product is non-degenerate.

Notice that

$$\{e^H;\ H = \{h_1 < \ldots < h_p\} \subseteq \{1,2,\ldots,n\}\}$$

is an orthonormal basis of $F^p(V)$.

The reference for this section is Flanders [2].

2.1.5 Star Operator

The following is based upon Flanders [2] but notation and argument have been changed.

When V is a vector space of dimension n with inner product, choose a fixed orientation of V and a basis which agrees with this orientation. Choose fixed

$$\lambda \in F^p(V)$$

and define a map

$$\varnothing : F^{n-p}(V) \to F^n(V)$$

by the rule

$$\varnothing(\mu) = \lambda \wedge \mu$$

When $\sigma = e_1 \wedge \ldots \wedge e_n$

is an orthonormal basis of $F^n(V)$, which has dimension 1, it follows that

$$\varnothing(\mu) = \lambda \wedge \mu$$

is a real multiple of σ. Say,

$$\varnothing(\mu) = \lambda \wedge \mu = f_\lambda(\mu)\sigma$$

where $f_\lambda(\mu)$ denotes the fact that the coefficient of σ depends on λ and μ.

Note

(i) <u>θ is linear</u>

since $\wedge$ is distributive over $+$.

(ii) f_λ <u>$: F^{n-p}(V) \to \mathbb{R}$ is linear</u> since

$$\begin{aligned} f_\lambda(\mu_1+\mu_2)\sigma &= \lambda \wedge (\mu_1+\mu_2) \\ &= \lambda \wedge \mu_1 + \lambda \wedge \mu_2 \\ &= f_\lambda(\mu_1)\sigma + f_\lambda(\mu_2)\sigma \end{aligned}$$

$$\therefore \quad f_\lambda(\mu_1+\mu_2) = f_\lambda(\mu_1) + f_\lambda(\mu_2)$$

Now if $f : W \to \mathbb{R}$ is a linear functional on vector space W then, to this particular functional f, there corresponds a unique vector $\underline{v}_f \in W$, depending only on f, such that

$$f(\underline{w}) = \langle \underline{v}_f, w \rangle$$

In fact, if

$$\underline{w} = (a_1, \ldots, a_n)$$

with respect to basis

$$\{\underline{e}_1, \ldots, \underline{e}_n\}$$

of W notice that

$$f(\underline{w}) = \langle \underline{v}_f, \underline{w} \rangle \text{ where } \underline{v}_f = (f(\underline{e}_1), \ldots, f(\underline{e}_n))$$

Applying this result to the linear functional

$$f_\lambda : F^{n-p}(V) \to \mathbb{R}$$

it is seen that

$$f_\lambda(\mu) = \langle v_{f_\lambda}, \mu \rangle$$

where v_{f_λ} depends solely on f_λ, that is, on λ.

Put

$$v_{f_\lambda} = *\lambda$$

to suggest this dependence.

$$\therefore \quad f_\lambda(\mu) = \langle *\lambda, \mu \rangle$$

Hence, $\theta(\mu) = \lambda \wedge \mu = f_\lambda(\mu)\sigma$

$$= \langle *\lambda, \mu \rangle \sigma$$

This defines a linear mapping,

$$* : F^p(V) \to F^{n-p}(V)$$

by $* : \lambda \to *\lambda$ given above.

<u>Proof</u>

that $*$ is linear.

Take

$a\lambda_1 + b\lambda_2 \in F^p(V)$ and use

$$\theta(\mu) = (a\lambda_1 + b\lambda_2) \wedge \mu$$

$$= f_{a\lambda_1 + b\lambda_2}(\mu)\sigma$$

$$= \langle *(a\lambda_1 + b\lambda_2), \mu \rangle \sigma \qquad \ldots(1)$$

However,

$$(a\lambda_1 + b\lambda_2) \wedge \mu$$

$$= a(\lambda_1 \wedge \mu) + b(\lambda_2 \wedge \mu)$$

$$= a\langle *\lambda_1, \mu\rangle\sigma + b\langle *\lambda_2, \mu\rangle\sigma \qquad \ldots(2)$$

Equations (1) and (2) imply that

$$*(a\lambda_1 + b\lambda_2) = a(*\lambda_1) + b(*\lambda_2)$$

By considering the effect of * on the generators of $F^p(V)$, certain important formulae emerge.

Suppose that $F'(V)$ has orthonormal basis $\{e_1,\ldots,e_n\}$. Let

$$H = \{1,\ldots,p\} \subset \{1,2,\ldots,n\}$$

$$K = \{K_{p+1},\ldots,K_n\} \subset \{1,2,\ldots,n\}$$

Put $\quad e^H = e_1 \wedge \ldots \wedge e_p \in F^p(V)$

$$e^K = e_{K_{p+1}} \wedge \ldots \wedge e_{K_n} \in F^{n-p}(V)$$

$$\therefore \quad e^H \wedge e^K = \langle *e^H, e^K\rangle\sigma$$

where $\sigma = e_1 \wedge \ldots \wedge e_n$. However $e^H \wedge e^K \neq 0$ implies that

$$H \cap K = \emptyset \text{ and } K = \{p+1,\ldots,n\}$$

which implies

$$\sigma = e^H \wedge e^K = \langle *e^H, e^K\rangle\sigma$$

But

$$e^H \wedge e^K \neq 0 \Rightarrow \langle *e^H, e^K \rangle \neq 0$$

$$\Rightarrow *e^H \text{ is a multiple of } e^K$$

$$\text{say : } *e^H = ce^K$$

$\therefore \quad \sigma = c\langle e^K, e^K \rangle \sigma$

which implies that

$$c\langle e^K, e^K \rangle = 1.$$

$\Rightarrow \quad c = \langle e^K, e^K \rangle = \pm 1$

Conclusion :

$$*e^H = \langle e^K, e^K \rangle e^K$$

On the other hand,

$$*e^K = (-1)^{p(n-p)} \langle e^H, e^H \rangle e^H$$

Proof

$$\varphi : e^H \longmapsto \mu \wedge e^H = \langle *\mu, e^H \rangle \sigma$$

where

$$\mu \wedge e^H \neq 0 \Rightarrow \mu = e_{p+1} \wedge \dots \wedge e_n$$

$$= e^K$$

$\therefore \quad e^K \wedge e^H = \langle *e^K, e^H \rangle \sigma$

Now using $e^K \wedge e^H = (-1)^{p(n-p)} e^H \wedge e^K$ by result (a) p.43, gives

$$(-1)^{p(n-p)} e^H \wedge e^K = \leq *e^K, e^H \geq \sigma$$

$$\Rightarrow \quad (-1)^{p(n-p)} \sigma = \leq *e^K, e^H \geq \sigma$$

$$\Rightarrow \quad (-1)^{p(n-p)} = \leq *e^K, e^H \geq$$

$$\Rightarrow \quad *e^K = c.e^H \text{ for some } c \in \mathbb{R}$$

$$\Rightarrow \quad (-1)^{p(n-p)} = c \leq e^H, e^H \geq$$

$$\Rightarrow \quad c = (-1)^{p(n-p)} \leq e^H, e^H \geq$$

because $\leq e^H, e^H \geq^2 = 1$

Therefore,

$$*e^K = (-1)^{p(n-p)} \leq e^H, e^H \geq e^H.$$

Another important result is the following

$$*(*e^H) = (-1)^{p(n-p)+(n-t)/2} e^H$$

where t is the signature of the inner product $\leq \ , \ \geq$.

<u>Proof</u>

Since $*e^H = \leq e^K, e^K \geq e^K$

then $*(*e^H) = \leq e^K, e^K \geq (*e^K)$ (* linear)

$$= (-1)^{p(n-p)} \leq e^H, e^H \geq . \leq e^K, e^K \geq e^H$$

by the result opposite. However,

$$\leq e^H, e^H \geq . \leq e^K, e^K \geq = \leq \sigma, \sigma \geq = (-1)^{\frac{n-t}{2}}$$

$$\therefore \quad *(*e^H) = (-1)^{p(n-p) + (n-t)/2} e^H, \text{ as required.}$$

By linearity, this last result may be generalized to any

$$\alpha \in F^p(V)$$

$$\therefore \quad **\alpha = (-1)^{p(n-p) + (n-t)/2}\alpha$$

The final result that is needed is

$$\alpha \wedge *\beta = \beta \wedge *\alpha = (-1)^{\frac{n-t}{2}} \leq\alpha, \beta\geq\sigma$$

$$\text{for } \alpha, \beta \in F^p(V)\,.$$

Proof

Merely show this to be true for basis vectors

$$\beta = e^H = e_{h_1} \wedge \ldots \wedge e_{h_p} \in F^p(V)$$

$$\therefore \quad *\beta = \leq e^K, e^K \geq e^K$$

where $H \cap K = \emptyset$ and $H \cup K = \{1, \ldots, n\}$

$$\therefore \quad \alpha \wedge *\beta = \leq e^K, e^K \geq \alpha \wedge e^K$$

But $\quad \alpha \wedge e^K \neq 0 \Rightarrow \alpha = e^H$

$$\therefore \quad \alpha \wedge *\beta = \leq e^K, e^K \geq \sigma$$

again use

$$\leq e^K, e^K \geq . \leq e^H, e^H \geq = (-1)^{\frac{n-t}{2}}$$

$$\therefore \quad \leq e^K, e^K \geq = (-1)^{(n-t)/2} \lll e^H, e^H \ggg$$

Hence, $\quad \alpha \wedge *\beta = (-1)^{(n-t)/2} \lll e^H, e^H \ggg \sigma$

$$= (-1)^{(n-t)/2} \lll \alpha, \beta \ggg \sigma$$

$$= (-1)^{(n-t)/2} \lll \beta, \alpha \ggg \sigma$$

$$= \beta \wedge *\alpha.$$

Applications

(1) In 4-space with the Lorentz metric the orthonormal basis is $\{dx_1, dx_2, dx_3, dt\}$ such that

$$<dx_i, dx_j> = \delta_{ij} \text{ with } <dx_i, dt> = 0$$

$$<dt, dt> = -1$$

$\therefore$ $<dx_i, dx_i> = 1$ for $i = 1,2,3$

and $<dt, dt> = -1$

means that this metric has signature 2.

$$\therefore \quad \frac{n-t}{2} = \frac{4-2}{2} = 1$$

$$\therefore \quad (-1)^{\frac{n-t}{2}} = -1.$$

Now consider $*(dx_i dt)$ for $i = 1,2,3$

If

$$H = \{i,4\} \text{ put } dx_i dt = e^H$$

$$\begin{aligned} \therefore \quad *(dx_i dt) &= *e^H \\ &= \leq e^K, e^K \geq e^K \\ &= \leq dx_j dx_k, dx_j dx_k \geq dx_j dx_k \\ &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} dx_j dx_k = dx_j dx_k \end{aligned}$$

$$\therefore \quad *(dx_i dt) = dx_j dx_k$$

where i,j,k are in cyclic order, can likewise show that

$$*(dx_j dx_k) = -dx_i dt$$

(2) In 3-space with the <u>Euclidean metric</u> let $F'(\mathbb{R}^3)$ have orthonormal basis $\{dx_1, dx_2, dx_3\}$ such that

$$\langle dx_i, dx_j \rangle = \delta_{ij} \qquad (i,j = 1,2,3)$$

Take $\quad \sigma = dx_1 dx_2 dx_3$

$$e^H = dx_1 \text{ and } e^K = dx_2 dx_3$$

$$\therefore \quad *(dx_1) = \langle e^K, e^K \rangle e^K$$

$$= \begin{vmatrix} \langle dx_2, dx_2 \rangle & \langle dx_2, dx_3 \rangle \\ \langle dx_3, dx_2 \rangle & \langle dx_3, dx_3 \rangle \end{vmatrix} dx_2 dx_3$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} dx_2 dx_3$$

$$= dx_2 dx_3$$

$$\therefore \quad *dx_1 = dx_2 dx_3$$

Similarly, $*dx_2 = dx_3 dx_1$

$$*dx_3 = dx_1 dx_2$$

Applying the result on page 58 we get also:

$$*dx_2 dx_3 = *(*dx_1) = dx_1$$

$$*dx_3 dx_1 = *(*dx_2) = dx_2$$

$$*dx_1 dx_2 = *(*dx_3) = dx_3.$$

As mentioned, the development of this section is based on that of Flanders [2] but much detail has been added, proofs supplied and notation amended.

DIFFERENTIAL FORMS AND EXTERIOR DERIVATIVES

The ideas of 2.1.1 to 2.1.5 will now be applied to the particular vector space M_m which is the tangent space to a C^∞ n-manifold M.

The main reference is to Hicks [4]

2.2.1 Tensors and Forms on a Differentiable Manifold

Let M be a C^∞ n-manifold. A p-co tensor at $m \in M$ is simply a p-co tensor on the vector space M_m (as defined on page 37).

Denote the set of all p-co tensors at $m \in M$ by

$$T^{o,p}(M_m).$$

If α is a member of this set then

$$\alpha : \underbrace{M_m \times \ldots \times M_m}_{p} \to \mathbb{R}$$

and α is p-linear.

For example, if M is a surface in $\mathbb{R}^3$, define

$$w : M_m \times M_m \to \mathbb{R}$$

by $w(\underline{a},\underline{b}) = a_1b_2 - a_2b_1$ where $a_1,a_2,b_1,b_2 \in \mathbb{R}$ and $\underline{a},\underline{b} \in M_m$ with

$$\underline{a} = (a_1,a_2)$$

$$\underline{b} = (b_1,b_2)$$

w measures the area spanned by the parallelogram two of whose sides are $\underline{a},\underline{b}$.

If $(x_1,\ldots,x_n)$ is a coordinate system about m, then M_m has basis

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}$$

so any $\underline{v}_m \in M_m$ has coordinates $(a_1,\ldots,a_n)$ w.r.t. this basis.
Define 1-co tensors dx_i by

$$dx_i(a_1,\ldots,a_n) = a_i$$

$$\therefore \quad dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}$$

This means that if M_m^* is the dual space of M_m then it has basis

$$\{dx_1,\ldots,dx_n\}.$$

By definition, then, $M_m^* = T^{0,1}(M_m)$.

As explained on page 37, $T^{o,p}(M_m)$ is endowed with a real vector space structure.

A <u>p-contra tensor at m</u> is just a p-linear function α such that

$$\alpha : \underbrace{M_m^* \times \ldots \times M_m^*}_{p} \to \mathbb{R}$$

The set of all p-contra tensors at m is also a real vector space which is denoted by $T^{p,o}(M_m)$. Define

$$T^{o,o}(M_m) = \mathbb{R}$$

Since contra tensors will not be used subsequently, no more need be said about them.

A <u>p-co tensor field</u> on an open subset U of a manifold M is a mapping with domain U such that

$$\alpha(m) = \text{a unique p-co tensor at m.}$$

The set of all such α is denoted by $T^{o,p}(U)$.

It is possible also to define in a similar way q-contra tensor fields on U but these will not be needed.

A <u>p-form at $m \in M$</u> is an alternating p-co tensor at m and $F^{o}(M_m)$ is the set of all p-forms at m. Again, define

$$F^{o}(M_m) = \mathbb{R}$$

Define α to be a <u>C^{∞} p-co tensor field</u> on an open subset of manifold M, if for any set $\{X_1,\ldots,X_p\}$ of C^{∞} vector fields on U, the function

$$\alpha(X_1,\ldots,X_p) : M \to \mathbb{R} \text{ given by}$$

$$\alpha(X_1,\ldots,X_p)(m) = \alpha_m(X_1(m),\ldots,X_p(m))$$

is a C^{∞} function on U, that is,

$$\alpha(X_1,\ldots,X_p) \in C^{\infty}(U,\mathbb{R}).$$

The foregoing definitions provide the necessary background for the definition of the most essential idea to this dissertation - namely, the idea of a <u>differential form</u>.

A C^{∞} p-form on an open set U of a differentiable manifold M is called a <u>differential p-form on U</u>. Such an object is simply an alternating, C^{∞} p-co tensor on U. The set of all differential p-forms on U is denoted by

$F^{p}(U)$ and in the special case of p = 0, define

$$F^{o}(U) = C^{\infty}(U,\mathbb{R})$$

so that a differential o-form on U is just a differentiable function on U.

Convention

A differential p-form on U will henceforth be referred to as a p-form on U - the understanding being that it will be C^∞.

When (U,ϕ) is a chart on M, we get coordinates $x_1,\ldots,x_n$ on U and a basis

$$\left.\frac{\partial}{\partial x_1}\right|_m, \ldots, \left.\frac{\partial}{\partial x_n}\right|_m \quad \text{of } M_m.$$

The dual space M_m^* has basis $\{dx_1,\ldots,dx_n\}$.

Any C^∞ vector field X on U can be written

$$X = a_1 \frac{\partial}{\partial x_1} + \ldots + a_n \frac{\partial}{\partial x_n}$$

where

$$a_i \in C^\infty(U,\mathbb{R})$$

$$\therefore \quad dx_i(X) = a_i$$

so each 1-co tensor field dx_i is C^∞ on U.

$$\therefore \quad dx_1,\ldots,dx_n$$

are differential 1-forms on U.

It transpires that any other 1-form on U may be expressed in terms of the dx_i.

Theorem

If $\alpha \in F'(U)$ then

$$\alpha = \sum_{i=1}^{n} f_i dx_i \text{ where } f_i = \alpha(\frac{\partial}{\partial x_i})$$

Proof

Take

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \in \mathfrak{X}(U)$$

where, of course, $a_i \in C^\infty(U, \mathbb{R})$

$$\therefore \quad \alpha(X) = \sum_{i=1}^{n} a_i \alpha(\frac{\partial}{\partial x_i}) \qquad \ldots(i)$$

Also,

$$(\sum_{i=1}^{n} f_i dx_i)X = \sum_{i=1}^{n} f_i a_i$$

$$\left(\text{since} \quad dx_i(\frac{\partial}{\partial x_j}) = \delta_{ij}\right)$$

$$= \sum_{i=1}^{n} a_i \alpha(\frac{\partial}{\partial x_i}) \qquad \ldots(ii)$$

since $f_i = \alpha(\frac{\partial}{\partial x_i})$

$$\therefore \quad \text{eqns (i) \& (ii)} \Rightarrow \alpha = \sum_{i=1}^{n} f_i dx_i$$

Conforming with the definition on page 43, define the exterior product on $F'(U)$ by

$$\alpha \wedge \beta = \sum_{\pi \in S_2} (\operatorname{sgn} \pi)(\alpha \otimes \beta) \circ \pi$$

for any $\alpha, \beta \in F'(U)$.

$$\therefore \quad (\alpha \wedge \beta)(X_1, X_2) = \alpha(X_1)\beta(X_2) - \alpha(X_2)\beta(X_1)$$

Note:

$$\wedge : F'(U) \times F'(U) \to F^2(U)$$

As an extension of the theorem on page 46 a basis of $F^o(U)$ is

$$\{dx_{i_1} \wedge \ldots \wedge dx_{i_p} \; ; \; 1 \le i_1 < \ldots < i_p \le n\}$$

Convention

Any member $\alpha \in F^p(U)$ will hence-forth be written in terms of this basis:

$$\alpha = \sum_{i_1 < i_2 < \ldots < i_p} a_{i_1, i_2, \ldots, i_p} dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

where $a_{i_1, \ldots, i_p} \in C^\infty(U, \mathbb{R})$.

Finally, the properties listed on page 43 extend to differential forms

i.e. $\quad \alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$

$\alpha \wedge \alpha = 0$ for any odd form

$\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$

and $\alpha \wedge (\beta_1+\beta_2) = \alpha \wedge \beta_1 + \alpha \wedge \beta_2$

where $\alpha \in F^p(U)$

$\beta, \beta_1, \beta_2 \in F^q(U)$

and $\gamma \in F^r(U)$.

2.2.2 Exterior Derivative

The motivation comes from the calculus of functions of the type:

$$\mathbb{R}^3 \xrightarrow{f} \mathbb{R}$$

If U is an open subset of $\mathbb{R}^3$ and if

$f : U \to \mathbb{R}$ is C^∞ then

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$\therefore \quad f \in F^0(U) \Rightarrow df \in F'(U) .$$

Again,

$$d(df) = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)dydz$$

$$+ \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)dzdx + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)dxdy$$

$$= 0$$

by virtue of the equality of mixed partial derivatives.

However, for a more general 1-form, say

$$\omega = Pdx + Qdy + Rdz$$

$$d\omega = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dzdy + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$

$$\neq 0$$

$$\therefore \quad \omega \in F'(U) \Rightarrow d\omega \in F^2(U).$$

The task is to decide whether such an operator d can be generalized to the case

$$d : F^p(U) \to F^{p+1}(U)$$

where U is an open set on a manifold M. The newly defined operator will naturally satisfy similar conditions to the one above.

These are:

$$\left.\begin{array}{ll} \text{(i)} & d(\omega+\eta) = d\omega + d\eta \\ \text{(ii)} & d(d\omega) = 0 \end{array}\right\} \quad \forall \omega, \eta \in F(U).$$

(iii) For any $f \in C^\infty(U, \mathbb{R})$ it is required that

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

where (V,ϕ) is a coordinate system on M such that

$$V \cap U \neq \emptyset$$

and $x_i = \upsilon_i \circ \phi$.

<u>Constructing d</u>

Suppose that ω is a p-form on U, then on $U \cap V$ we have:

$$\omega = \sum_{i_1 < \dots < i_p} a_{i_1,\dots,i_p} dx_{i_1} \wedge \dots \wedge dx_{i_p}$$

Now define $d : F^p(U) \to F^{p+1}(U)$ by

$$d\omega = \sum_{j=1}^{n} \sum_{i_1<\ldots<i_p} \frac{\partial a_{i_1,\ldots,i_p}}{\partial x_j} (dx_j) \wedge \left(dx_{i_1} \wedge \ldots \wedge dx_{i_p}\right)$$

and show that properties (i), (ii), (iii) above are satisfied.

(i) follows directly from the linearity of $\frac{\partial}{\partial x_j}$

(ii) follows by using $dx_i \wedge dx_j = -dx_j \wedge dx_i$

together with

$$\frac{\partial^2 a}{\partial x_i \partial x_j} - \frac{\partial^2 a}{\partial x_j \partial x_i} = 0.$$

(iii) $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$

coming straight from the definition of d.

A consequence of the definition of d is a fourth property:

(iv) If $\lambda \in F^p(U)$ and $\mu \in F^q(U)$ then

$$d(\lambda \wedge \mu) = (d\lambda) \wedge \mu + (-1)^{\deg\lambda} \lambda \wedge d\mu.$$

Proof

Take $\lambda = \sum_H a_H dx^H$

$\mu = \sum_K b_K dx^K$

where

$H = \{h_1, \ldots, h_p\}$

$K = \{k_1, \ldots, k_q\}$

are ordered subsets of $\{1,2,\ldots,n\}$ and

$dx^H = dx_{h_1} \wedge \ldots \wedge dx_{h_p}$

etc.

$$\therefore \quad d(\lambda \wedge \mu) = \sum_H \sum_K d(a_H b_K dx^H \wedge dx^K)$$

$$= \sum_i \sum_H \sum_K \frac{\partial(a_H b_K)}{\partial x_i} dx_i dx^H dx^K$$

$$= \sum_i \sum_H \sum_K b_K \frac{\partial a_H}{\partial x_i} dx_i dx^H dx^K + \sum_i \sum_H \sum_K a_H \frac{\partial b_k}{\partial x_i} dx_i dx^H dx^K$$

$$= (\sum_i \sum_H \frac{\partial a_H}{\partial x_i} dx_i dx^H) \wedge \sum_K b_K dx^K + (-1)^{\deg\lambda} (\sum_H a_H dx^H) \wedge \sum_i \sum_K \frac{\partial b_K}{\partial x_i} dx_i dx^K$$

$$= d\lambda \wedge \mu + (-1)^{\deg\lambda} (\lambda \wedge d\mu).$$

The question about the uniqueness of d is answered as follows:

Assume that

$$d' : F^p(U) \to F^{p+1}(U)$$

is another map satisfying conditions (i) to (iv). Show

$d = d'$ as follows.

If $\omega \in F^p(U)$, say

$$\omega = \sum_H a_H dx^H$$

then

$$\begin{aligned} d'\omega &= \sum_H d'(a_H dx^H) && \text{by (i)} \\ &= \sum_H \{d'a_H \wedge dx^H + a_H d(dx^H)\} && \text{by (iv)} \\ &= \sum_H d'a_H \wedge dx_H && \text{by (ii)} \\ &= \sum_i \sum_H \frac{\partial a_H}{\partial x_i} d_x \wedge dH_H && \text{by (iii)} \\ &= d\omega \text{ by definition of d.} \end{aligned}$$

Hence, $d = d'$

and the unique mapping

$$d : F^p(U) \to F^{p+1}(U)$$

is called the <u>exterior derivative</u>.

The reference is to Flanders [2]

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2.2.3 Mappings

The following ideas are not subsequently used but are noted merely to give a more complete picture of the basic ideas of different forms on a manifold.

Let M and N be C^∞ manifolds and suppose that

$$\Psi : M \to N \text{ is a } C^\infty \text{ mapping.}$$

Consequently,

$$\Psi_* : TM \to TN$$

and in particular

$$(\Psi_*)_m : M_m \to N_{\psi(m)}$$

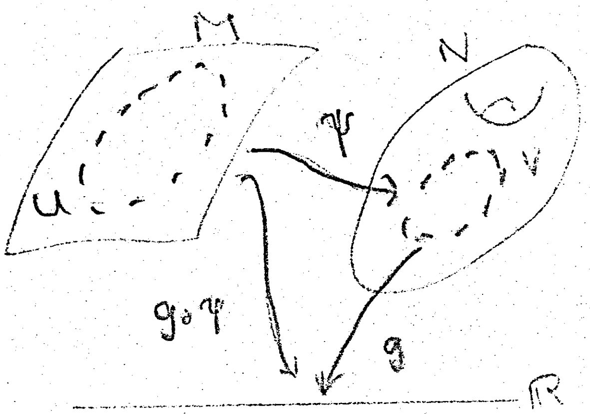
What relationships are induced by Ψ_* between vector fields and forms on M and vector fields and forms on N?

Generally, a vector field X on M can be transferred to N by Ψ_* only when Ψ is a diffeomorphism. In this case, Ψ^{-1} exists and the sequence

$$N \xrightarrow{\Psi^{-1}} M \xrightarrow{X} TM \xrightarrow{\Psi_*} TN$$

depicts the C^∞ vector field $\Psi_* \circ X \circ \Psi^{-1}$ on N.

Again, if f is a real-valued C^∞ function on M (a 0-form), then there is, in general, no way of moving f over to N but the reverse <u>is</u> true.



that is, if $g \in F^0(V)$ where $V \subseteq N$ is open, then $g \circ \Psi \in F^0(U)$ since $g \circ \Psi$ is C^∞ on

$$U = \Psi^{-1}(V)$$

In general, if $\omega \in F^p(V)$ then Ψ induces a p-form on

$$U = \Psi^{-1}(V)$$

denoted by $\Psi^*\omega$ called the <u>pull-back of ω</u>. It is given by

$$\Psi^*\omega = \omega \circ \Psi_*$$

The definition is made explicit as follows.

Suppose that $n = \Psi(m)$ and that $\underline{v}_1, \ldots, \underline{v}_p \in N_n$ with $\Psi_*(\underline{u}_i) = \underline{v}_i$ for $i = 1, \ldots, p$ with $\underline{u}_i \in M_m$.

Since

$$\omega \in F^p(V) \text{ then } \omega_n : (N_n)^p \to \mathbb{R}$$

by

$$\omega_n(\underline{v}_1, \ldots, \underline{v}_p) \in \mathbb{R} \text{ so } \Psi^*\omega \in F^p(U)$$

is defined by:

$$(\Psi^* \omega)_m : (M_m)^p \to \mathbb{R}$$

where

$$(\Psi^* \omega)_m(\underline{u}_1, \ldots, \underline{u}_p)$$

$$= \omega_{\Psi(m)}(\Psi_*(\underline{u}_1), \ldots, \Psi_*(\underline{u}_p))$$

$$= \omega_n(\underline{v}_1, \ldots, \underline{v}_p).$$

The following properties are listed and proved in Flanders [2] and Hicks [4]

(1) If ω is C^∞ on V then $\Psi^*\omega$ is C^∞ on $\Psi^{-1}(V)$

(2) $\Psi^*(\alpha_1 + \alpha_2) = \Psi^*(\alpha_1) + \Psi^*(\alpha_2)$

$\Psi^*(\alpha_1 \otimes \alpha_2) = \Psi^*(\alpha_1) \otimes \Psi^*(\alpha_2)$

$\Psi^*(\alpha_1 \wedge \alpha_2) = \Psi^*(\alpha_1) \wedge \Psi^*(\alpha_2)$

these are proved in a more general context on pages 49 and 50.

(3) $d \circ \Psi^* = \Psi^* \circ d.$

(4) $(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^*$

where $M \xrightarrow{\Psi} N \xrightarrow{\Phi} P.$

2.2.4 Classical Vector Analysis

In Euclidean 3-space with its basic $\{\underline{i},\underline{j},\underline{k}\}$ the identification

$$\underline{i} \leftrightarrow \frac{\partial}{\partial x_1}$$

$$\underline{j} \leftrightarrow \frac{\partial}{\partial x_2}$$

$$\underline{k} \leftrightarrow \frac{\partial}{\partial x_3}$$

is often made via the notion of directional derivative. Compatible with this notation is the convention of regarding $\mathbb{R}^3$ itself as being a 3-manifold whose atlas consists of the single chart given by the identity map

$$i : \mathbb{R}^3 \to \mathbb{R}^3 \text{ with coordinate system}$$

$\{x_1,x_2,x_3\}$.

The module $\chi(\mathbb{R}^3)$ of all C^∞ vector fields over $C^\infty(\mathbb{R}^3,\mathbb{R})$ is generated by $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\}$.

Corresponding to this is the module $F^1(\mathbb{R}^3)$ over $C^\infty(\mathbb{R}^3,\mathbb{R})$ and this is generated by

$$\{dx_1,dx_2,dx_3\}$$

where

$$dx_i(\frac{\partial}{\partial x_j}) = \delta_{ij}$$

This correspondence is made explicit by considering the Euclidean inner product on $\mathbb{R}^3$ denoted by

$$< , >,$$

That is, there is a natural isomorphism $f : \chi(\mathbb{R}^3) \to F'(\mathbb{R}^3)$ defined by

$$f(X) = f_X \in F'(\mathbb{R}^3)$$

where $\quad f_X(Y) = \langle X, Y \rangle.$

Proof

f is 1-1.

$$f(X_1) = f(X_2) \iff f_{X_1} = f_{X_2}$$

$$\iff \langle X_1, Y\rangle = \langle X_2, Y\rangle$$

for all Y.

$$\Rightarrow X_1 = X_2$$

$\therefore \quad$ f is 1-1.

f is linear

Since < , > is bilinear.

f is onto

If $\{\underline{e}_1,\underline{e}_2,\underline{e}_3\}$ is a basis of $\chi(\mathbb{R}^3)$ let

$$<\underline{e}_i,\underline{e}_j> = g_{ij} \qquad ij = 1,2,3$$

so $g_{ij} \in C^\infty(\mathbb{R}^3,\mathbb{R})$.

If $\{\phi_1,\phi_2,\phi_3\}$ is the dual basis then

$$f(\underline{e}_i) = \sum_{j=1}^{3} g_{ij}\phi_j$$

and, when basis $\{\underline{e}_1,\underline{e}_2,\underline{e}_3\}$ is orthonormal then

$$g_{ij} = \delta_{ij}$$

$$\therefore \qquad f(\underline{e}_i) = \phi_i$$

If $\omega \in F'(\mathbb{R}^3)$ is given by

$$\omega = \sum_{i=1}^{3} a_i\phi_i$$

then

$$\omega = f(X) \text{ where } X = \sum_{i=1}^{3} a_i\underline{e}_i$$

Hence, f is onto.

Note that $f(\frac{\partial}{\partial x_i}) = dx_i$ so that if

$$W = \sum_{i=1}^{3} W_i \frac{\partial}{\partial x_i} \text{ where } W_i \in C^\infty(\mathbb{R}^3, \mathbb{R})$$

then

$$W \in \chi(\mathbb{R}^3) \text{ and } f(W) = \sum_{i=1}^{3} W_i dx_i = \omega$$

where ω is a dual 1-form of W. This notion is central to the applications of Chapter Three of this essay.

Apart from the isomorphism of $\chi(\mathbb{R}^3)$ with $F'(\mathbb{R}^3)$ as modules, it transpires that the classical vector analysis of $\chi(\mathbb{R}^3)$ has a complete analogue in $F'(\mathbb{R}^3)$.

This analogy is not drawn merely for the sake of change, because it can be shown that many of the classical theorems, for example, Stokes Theorem, are more succinctly expressed in $F'(\mathbb{R}^3)$ and are more amenable to generalization.

As a first step in vindicating the above claim, analogues of the familiar operators curl, div and grad will be constructed in $F'(\mathbb{R}^3)$. What follows is an extension and a development of the ideas given on page 150 of Warner [7].

Curl of a 1-form

If $W = \sum_{i=1}^{3} W_i \frac{\partial}{\partial x_i} \in \chi(\mathbb{R}^3)$

then, as per the classical definition,

$$\text{curl } W = \left(\frac{\partial W_3}{\partial x_2} - \frac{\partial W_2}{\partial x_3}\right)\frac{\partial}{\partial x_1} + \left(\frac{\partial W_1}{\partial x_3} - \frac{\partial W_3}{\partial x_1}\right)\frac{\partial}{\partial x_2} + \left(\frac{\partial W_2}{\partial x_1} - \frac{\partial W_1}{\partial x_2}\right)\frac{\partial}{\partial x_3}$$

which dual 1-form

$$\left(\frac{\partial W_3}{\partial x_2} - \frac{\partial W_2}{\partial x_3}\right)dx_1 + \left(\frac{\partial W_1}{\partial x_3} - \frac{\partial W_3}{\partial x_1}\right)dx_2 + \left(\frac{\partial W_2}{\partial x_1} - \frac{\partial W_1}{\partial x_2}\right)dx_3$$

However, the dual of W is the 1-form

$$\omega = \sum_{i=1}^{3} W_i dx_i$$

and

$$d\omega = \left(\frac{\partial W_3}{\partial x_2} - \frac{\partial W_2}{\partial x_3}\right)dx_2 dx_3 + \left(\frac{\partial W_1}{\partial x_3} - \frac{\partial W_3}{\partial x_1}\right)dx_3 dx_1 + \left(\frac{\partial W_2}{\partial x_1} - \frac{\partial W_1}{\partial x_2}\right)dx_1 dx_2$$

The comparison to make is that of $d\omega$ with the dual 1-form of curl W given overleaf. But use of the formulae on page 61 gives

$$*d\omega = \left(\frac{\partial W_3}{\partial x_2} - \frac{\partial W_2}{\partial x_3}\right)dx_1 + \left(\frac{\partial W_1}{\partial x_3} - \frac{\partial W_3}{\partial x_1}\right)dx_2 + \left(\frac{\partial W_2}{\partial x_1} - \frac{\partial W_1}{\partial x_2}\right)dx_3$$

This motivates the <u>definition</u>

$$\text{curl } \omega = *d\omega$$

Vector Product of 1-forms

Consider $\alpha, \beta \in F'(\mathbb{R}^3)$ where

$$\alpha = \sum_{i=1}^{3} a_i dx_i \text{ and } \beta = \sum_{j=1}^{3} b_j dx_j$$

Comparison with the definition of the vector product of the corresponding vector fields A, $B \in \chi(\mathbb{R}^3)$ suggests that

$$\alpha \times \beta = \Sigma (a_j b_k - a_k b_j) dx_i$$

$$= \begin{vmatrix} dx_1 & dx_2 & dx_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Note, however, that

$$\alpha \wedge \beta = \begin{vmatrix} dx_2 dx_3 & dx_3 dx_1 & dx_1 dx_2 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Hence define

$$\alpha \times \beta = *(\alpha \wedge \beta)$$

by again using the formulae of p.61. This is the vector product of α with β.

Inner Product of 1-forms

The isomorphism between $\chi(\mathbb{R}^3)$ and $F'(\mathbb{R}^3)$ provides an inner product of 1-forms defined by

$$\leq\alpha,\beta\geq = \sum_i a_i b_i \qquad \text{where } \alpha = \sum_i a_i dx_i$$

$$\beta = \sum_i b_i dx_i$$

$$= \sum_i a_i b_i *(dx_1 dx_2 dx_3)$$

$$= *\{(\Sigma\, a_i dx_i) \wedge (\Sigma\, b_j dx_h dx_k)\}$$

$$= *\{(\Sigma\, a_i dx_i) \wedge *(\Sigma\, b_j dx_j)\}$$

$$= *(\alpha \wedge *\beta)$$

This inner product can be alternatively described, therefore, as

$$\leq\alpha,\beta\geq = *(\alpha \wedge *\beta)$$

where, in fact,

$$\leq\ ,\ \geq\ =\ <\ ,\ >\circ f$$

and f and < , > are as defined on page 79.

Divergence of a 1-form

For a vector field

$$\sum_{i=1}^{3} a_i \frac{\partial}{\partial x_i} .$$

The traditional definition of divergence is

$$\text{div } A = \sum_{i=1}^{3} \frac{\partial a_i}{\partial x_i}$$

If

$$\alpha = \sum_{i=1}^{3} a_i dx_i$$

is the dual 1-form of A then the fact that

$$*(d*\alpha) = \sum \frac{\partial a_i}{\partial x_i}$$

suggests the definition

$$\underline{\text{div } \alpha = *(d*\alpha)}$$

Gradient of a 0-form

In the classical sense, a function

$$\phi : \mathbb{R}^3 \to \mathbb{R}$$

has gradient

$$\text{grad } \phi = \sum_{i=1}^{3} \frac{\partial \phi}{\partial x_i} \frac{\partial}{\partial x_i}$$

and this has dual 1-form

$$\sum_{i=1}^{3} \frac{\partial\phi}{\partial x_i} dx_i = d\phi$$

$\therefore$ the 0-form ϕ has gradient

$$\text{grad } \phi = d\phi = \Sigma \frac{\partial\phi}{\partial x_i} dx_i$$

Using the above definitions, some classical theorems of vector analysis will be alternatively expressed and proved.

(i) $\text{curl}(\text{grad } \phi) = \text{curl}(d\phi)$

$$= *d(d\phi)$$

$$= *(d^2\phi) = 0$$

(ii) $\text{div}(\text{curl } \omega) = \text{div}(*d\omega)$

$$= *d(**d\omega)$$

$$= *d^2\omega = 0$$

(iii) If $\underline{A}$ is a C^∞ vector field on a compact 2-manifold, V, embedded in $\mathbb{R}^3$ with boundary ∂V then

$$\int_V \text{div } \underline{A} \, dx_1 dx_2 dx_3 = \int_{\partial V} A_1 \, dx_2 dx_3 + A_2 \, dx_3 dx_1 + A_3 \, dx_1 dx_2$$

This is the <u>divergence theorem</u>

Let $\omega = A_1\, dx_2 dx_3 + A_2\, dx_3 dx_1 + A_3\, dx_1 dx_2$

then the above equation becomes

$$\int_V d\omega = \int_{\partial V} \omega$$

where ω is a 2-form.

However, if

$$\alpha = \sum_{i=1}^{3} A_i\, dx_i$$

is the dual 1-form,

$$\int_V \operatorname{div} \alpha = \int_{\partial V} {*\alpha}$$

is an alternative expression of the divergence theorem.

(iv) Let S be a 2-manifold in $\mathbb{R}^3$ with boundary C (a closed curve) then the classical version of <u>Stoke's Theorem</u> is the following:

$$\int_S \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) dx_2 dx_3 + \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) dx_3 dx_1 + \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right) dx_1 dx_2$$

$$= \int_C A_1\, dx_1 + A_2\, dx_2 + A_3\, dx_3 \qquad \ldots(1)$$

If $\alpha = \Sigma A_i\, dx_i$ is the dual 1-form of A

$$\left.\begin{array}{ll} \text{then} & \displaystyle\int_S d\alpha = \int_C \alpha \\ \underline{\text{or}} & \displaystyle\int_S {*\text{curl}}\ \alpha = \int_{\partial S} \alpha \end{array}\right\} \text{ are both equivalent to (1) above.}$$

Generalizations

If M is a C^∞ n-manifold and if U is an open subset of M, then define

$$\text{grad} : F^0(U) \to F'(U)$$

by

$$\text{grad}\ \phi = d\phi$$

called the gradient of the 0-form ϕ.

When

$$\alpha \in F'(U),$$

defin its divergence to be

$$\text{div}\ \alpha = *(d{*}\alpha) \in F'(U)$$

The concepts of vector product and curl cannot be generalized.

Finally, the Generalized Stoke's Theorem is

$$\int_{\partial \underline{C}} \omega = \int_{\underline{C}} d\omega$$

where ω is a p-form on M and $\underline{C}$ is a (p+1)-chain on M. This, however, invokes the idea of a simplicial complex on M and the description of such ideas is not within thescope of this essay.

2.2.5 Closed and Exact Forms

One of the basic properties of the exterior derivative d is that

$$d^2\omega = d(d\omega)$$
$$= 0$$

for any p-form ω defined on an open set U of a manifold M.

It may happen that, for some ω, we have

$$d\omega = 0$$

and such a form is said to be closed.

Given a p-form, ω, it may be possible to find a (p-1)-form η such that

$$\omega = d\eta$$

and in this case ω is said to be exact. It is certainly true that any exact form is closed because

$$d(d\eta) = 0$$

However, the converse is not necessarily true. The following example is the solution to exercise 15 of page 159 of Warner [7].

Let $\omega = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$

be defined on $\mathbb{R}^2 - \{0\}$

$$\therefore \qquad d\omega = \frac{y^2-x^2}{(x^2+y^2)^2}\,dydx + \frac{y^2-x^2}{(x^2+y^2)^2}\,dxdy$$

$$= 0 \text{ since } dydx = -dxdy$$

$\therefore$ ω is closed.

If however, ω is exact then there is a C^∞ 0-form f such that

$$\omega = df$$

so when C is any closed curve in $\mathbb{R}^2 - \{0\}$ we get

$$\int_C \omega = \int_C df$$

$$= \int_{\partial C} f = 0$$

If C is the unit circle, we have

$$\int_C \omega = 2\pi$$

by direct integration using

$$x = \cos\phi,\ y = \sin\phi$$

This is a contradiction so ω is not exact.

Poincare's Lemma gives the condition under which a closed form is exact:

'Let U be an open subset of $\mathbb{R}^n$ which is contractible to a point. If ω is a p-form on U such that $d\omega = 0$, then

$$\exists \eta \in F^{p-1}(U);\ \omega = d\eta\text{'}$$

see Hicks [4]

This is stated in Spivak [6] for a general manifold M as follows:

'If a manifold M is contractible to a point then every closed form ω on M is exact.'

The concept of contractibility to which Spivak refers is explained as follows:

Manifold M is contractible to a point $m_0 \in M$ if there exists a C^∞ function

$$H : M \times [0,1] \rightarrow M$$

such that

$$H(m,1) = m$$

$$H(m,0) = m_0$$

$$\forall m \in M.$$

Some of the preceding ideas may be interpreted in terms of classical vector analysis as follows.

(i) 'A vector field whose curl is zero is the gradient of some scalar function

<u>i.e.</u> $\text{curl } \underline{V} = 0 \Rightarrow \underline{V} = \text{grad } \phi$

If ω is the dual 1-form of $\underline{V}$ then

$$\text{curl } \omega = {*d\omega} = 0$$

so that $d\omega = 0$ with $\omega = d\phi$. Field $\underline{V}$ is called <u>irrotational.</u>

(ii) 'A vector field whose divergence is zero, is the curl of some other vector field

<u>i.e.</u> $\text{div } \underline{V} = 0 \Rightarrow \underline{V} = \text{curl } \underline{W}$

If r is the dual 1-form of $\underline{V}$ then

$$d(*r) = 0 \Leftrightarrow \text{div } \underline{V} = 0$$

$$\therefore \quad \omega \; ; \; d\omega = *r$$

<u>i.e.</u> $r = {*d\omega} = \text{curl } \omega$

Field $\underline{V}$ in this case is called <u>solenoidal</u>

These classical theorems usually assume that the vector fields $\underline{V}$ are defined on contractible regions of $\mathbb{R}^3$.

2.3.1 Summary

The most general idea discussed in this Chapter has been that of a tensor product on an n-dimensional vector space. Particular kinds of tensors called p-forms were considered. This leads to the idea of an exterior algebra over V as being the vector space F(V) formed by all p-forms ($p \leq n$).

When V is chosen to be the tangent space M_m at a point m of a C^∞ manifold M then $F^p(U)$ was defined to be the set of all C^∞ p-forms called differential forms over the open set U containing m.

Two important maps were introduced namely,

$$d : F^p(U) \to F^{p+1}(U)$$

and

$$* ; F^p(U) \to F^{n-p}(U)$$

and, by use of these, some well known results from classical vector analysis are expressed in the language of differential forms.

Specifically, by virtue of the Euclidean metric on $\mathbb{R}^n$, there is a natural isomorphism between the set $\chi(\mathbb{R}^n)$ of all C^∞ vector fields on $\mathbb{R}^n$ and the set $F'(\mathbb{R}^n)$ of all 1-forms on $\mathbb{R}^n$. Depending on the values of p and n, some or all of the functions

inner product

vector product

div

grad

curl

have their analogues in $F^p(\mathbb{R}^n)$.

The set $F'(\mathbb{R}^n)$ is called the set of <u>dual 1-forms</u>.

In terms of exterior product $\wedge$

star operator $*$

exterior derivative d

the 'isomorphism' between $\chi(\mathbb{R}^3)$ and $F'(\mathbb{R}^3)$ is summarized as follows:

INNER PRODUCT	$\leq\alpha,\beta\geq\ = *(\alpha \wedge *\beta)$
VECTOR PRODUCT	$\alpha \times \beta = *(\alpha \wedge \beta)$
DIVERGENCE	$\text{div}\ \alpha = *(d*\alpha)$
CURL	$\text{curl}\ \alpha = *d\alpha$

where α, β are 1-forms. For a 0-form <u>define</u>

$$\text{grad}\ \phi = d\phi \in F'(\mathbb{R}^3)$$

Generalizing these to the case of a pseudo-Riemannian, C^∞ n-manifold M with open subset U we have:

INNER PRODUCT	$\leq\alpha,\beta\geq\ = *(\alpha \wedge *\beta)$
DIVERGENCE	$\text{div}\ \alpha = *d(*\alpha)$
GRADIENT	$\text{grad}\ \phi = d\phi$

Finally, the classical Stoke's and Divergence Theorems are both related an are particular versions of the Generalized Stokes Theorem.

$$\int_{\underline{\partial C}} \omega = \int_{\underline{C}} d\omega$$

Chapter 3

DUAL FORMS, ELECTROMAGNETIC THEORY AND THE LONDON EQUATIONS

3.1.1 Background

The work of Chapter 2 suggests that vector analysis can be replaced by differential forms as a tool for solving problems in, say, electromagnetic theory. The aim of this chapter is to show how this may be done and specific reference is made to Maxwell's equations.

The principal vector fields involved are

(i) Electric field $\underline{E}$

(ii) Magnetic field $\underline{H}$

(iii) Magnetic flux density $\underline{B}$ which is given by

$$\underline{B} = \eta_o \underline{H}$$

where η_o is constant.

(iv) Electric current density $\underline{J}$ which is defined by

$$\underline{E} = p^1 \underline{J}$$

where p^1 is the resistivity of the medium.

(v) Electric flux density $\underline{D}$ given by $\underline{D} = \varepsilon \underline{E}$ where ε is a constant which is characteristic of the medium in which the fields are present.

In the magnetic field, the primary vector is the magnetic flux density $\underline{B}$ which is defined in terms of the mechanical force on a current loop. The magnetic circuit law, defined later, suggests the introduction of a secondary vector field $\underline{H}$ which is associated with the currents that <u>cause</u> the field whereas $\underline{B}$ is associated with the <u>effect</u> of the field. In fact, the equation

$$\underline{B} = \eta_o \underline{H}$$

describes the torque on a dipole or current loop due to the field $\underline{H}$.

Similarly, in the electric field, the primary vector is the electric force field $\underline{E}$ which is defined in terms of the mechanical force on a unit charge. Gauss' theorem suggests the introduction of electric flux density $\underline{D}$ as the secondary vector which expresses the idea of unit emanation of flux from each unit of charge. Specifically, since

$$\iint_{S} \underline{E} \cdot \underline{dS} = 4\pi\lambda \; Q$$

where Q is the total charge and λ is a constant which is characteristic of the material, then

$$\iint_{S} \underline{D} \cdot \underline{dS} = Q$$

where $\underline{D} = \dfrac{1}{4\pi\lambda} \underline{E}$ and $\varepsilon = \dfrac{1}{4\pi\lambda}$

giving $\underline{D} = \varepsilon \, \underline{E}$

where ε is a constant which is characteristic of the medium.

The Maxwell equations emanate from four physical laws as follows:

(a) Law of Magnetic Induction

(or otherwise Faraday's Law)

'Every changing magnetic field has a corresponding electric field; the line integral of $\underline{E}$ round any closed circuit C is related to the flux φ across C by

$$\int_C E_s ds = -\frac{d\phi}{dt}, \text{ where } \phi = \iint_s \underline{B} \cdot \underline{dS}$$

and S is the surface spanning closed curve C. In vector notation this can be expressed as

$$\text{curl } \underline{E} = -\frac{1}{C}\frac{\partial B}{\partial t}$$

where C is the speed of light.

(b) <u>Generalised Magnetic Circuit Law</u>

'The line integral of $\underline{H}$ round any closed path is equal to the total current linked with that path'. Expressed classically as

$$\int_C H_s ds = I + \frac{d\Psi}{dt} \quad \text{where } \Psi = \iint_s \underline{D} \circ \underline{dS}$$

and Ψ is the electric flux. Alternatively,

$$\text{curl } \underline{H} = \frac{4\pi}{C}\underline{J} + \frac{1}{C}\frac{\partial D}{\partial t}$$

(c) <u>Gauss' Theorem</u>

'The total flux of electric force out of any closed surface is equal to 4π times the total charge enclosed by that surface'

i.e. $\int_s \underline{E} \cdot \underline{dS} = 4\pi \circ Q$

or $\text{div } \underline{D} = 4\pi p$.

where p is the charge density and Q is the total charge.

(d) Solenoidal Property

'The net magnetic flux of $\underline{B}$ out of any closed surface is zero'.

That is,

$$\iint_S \underline{B} \circ d\underline{S} = 0 \text{ or } \operatorname{div} \underline{B} = 0$$

Summarising, we have the Maxwell equations

$$\operatorname{curl} \underline{E} = -\frac{1}{C}\frac{\partial B}{\partial t}$$

$$\operatorname{curl} \underline{H} = \frac{4\pi}{C}\underline{J} + \frac{1}{C}\frac{\partial \underline{D}}{\partial t}$$

$$\operatorname{div} \underline{D} = 4\pi p$$

$$\operatorname{div} \underline{B} = 0$$

It is worth noting that the operators div, grad and curl have the following physical interpretation:

The divergence of a vector field $\underline{A}$ at point P measures the outward flux per unit volume at P.

The gradient of a scalar field Φ, is denoted by $\nabla\Phi$ and $\nabla\Phi.\underline{a}$ gives the rate of charge of Φ in the direction of the unit vector $\underline{a}$.

The curl of a vector field is a measure of the rotational properties of that field.

The reference for much of this chapter is Flanders [4].

3.1.2 Maxwell's Equations

From the previous section these are

(i) $\text{curl } \underline{E} = -\frac{1}{C} \frac{\partial \underline{B}}{\partial t}$

(ii) $\text{curl } \underline{H} = \frac{4\pi}{C} \underline{J} + \frac{1}{C} \frac{\partial \underline{D}}{\partial t}$

(iii) $\text{div } \underline{D} = 4\pi \, p$

(iv) $\text{div } \underline{B} = 0$

The fields are given by

$$\underline{E} = E_1 \underline{i} + E_2 \underline{j} + E_3 \underline{k}$$

$$\underline{H} = H_1 \underline{i} + H_2 \underline{j} + H_3 \underline{k}$$ with similar

equations defining $\underline{D}$, $\underline{B}$ and $\underline{J}$.

On page 80 it is proved that corresponding to every vector field on $\mathbb{R}^3$, there is a dual 1-form. Consequently if we regard $\underline{E}$, $\underline{H}$, $\underline{B}$ and $\underline{D}$ as vector fields in any of the senses described in 1.3.1, then they too have dual 1-forms. These 1-forms are as follows:

$$\omega_E = E_1 dx_1 + E_2 dx_2 + E_3 dx_3$$

$$\omega_H = H_1 dx_1 + H_2 dx_2 + H_3 dx_3$$

$$\omega_D = D_1 dx_1 + D_2 dx_2 + D_3 dx_3$$

$$\omega_B = B_1 dx_1 + B_2 dx_2 + B_3 dx_3$$

$$\omega_J = J_1 dx_1 + J_2 dx_2 + J_3 dx_3$$

Using the equations of 2.2.4 which define the div and curl of a 1-form ω, we have

$$\text{curl } \omega = *d\omega \text{ and } \text{div } \omega = *d(*\omega)$$

So Maxwell equations become:

$$*d\omega_E = -\frac{1}{C}\ \frac{\partial\omega_B}{\partial t} = -\frac{1}{C}\dot{\omega}_B$$

$$*d\omega_H = \frac{4\pi}{C}\omega_J + \frac{1}{C}\dot{\omega}_D$$

$$*d(*\omega_D) = 4\pi p$$

$$*d(*\omega_B) = 0$$

Now use the equation

$$**\alpha = (-1)^{p(n-p)+\frac{(n-t)}{2}}\alpha \quad \text{from p.58}$$

with p=2, n=3, t=3

$$\therefore\ **d\omega_E = d\omega_E$$

$$**d\omega_H = d\omega_H \qquad \text{etc., etc.}$$

and the above equations become

$$(v)'\ d\omega_E = -\frac{1}{C} * \mathring{\omega}_B$$

$$(vi)'\ d\omega_H = \frac{4\pi}{C} * \omega_J + \frac{1}{C} * \dot{\omega}_D$$

$$\text{(vii)}' \quad d(*\omega_D) = 4\pi p \; dx_1 \; dx_2 \; dx_3$$

$$\text{(viii)}' \quad d(*\omega_B) = 0$$

Note that

$$\dot{\omega}_B = \dot{B}_1 \; dx_1 + \mathring{B}_2 \; dx_2 + \dot{B}_3 \; dx_3$$

$$\dot{\omega}_D = \dot{D}_1 \; dx_1 + \mathring{D}_2 \; dx_2 + \dot{D}_3 \; dx_3$$

where the dot represents differentiation with respect to t.

If we put $\theta_B = *\omega_B$

$\theta_D = *\omega_D$

$\theta_J = *\omega_J$

where $\theta_B = B_1 \; dx_2 \wedge dx_3 + B_2 \; dx_3 \wedge dx_1 + B_3 \; dx_1 \wedge dx_2$

$$\theta_D = D_1 \; dx_2 \wedge dx_3 + D_2 \; dx_3 \wedge dx_1 + D_3 \; dx_1 \wedge dx_2$$

$$\theta_J = J_1 \; dx_2 \wedge dx_3 + J_2 \; dx_3 \wedge dx_1 + J_3 \; dx_1 \wedge dx_2$$

we get

$$\text{(v)} \quad d\omega_E = -\frac{1}{C} \dot{\theta}_B$$

$$\text{(iv)} \quad d\omega_H = \frac{4\pi}{C} \theta_J + \frac{1}{C} \mathring{\theta}_D$$

$$\text{(vii)} \quad d\theta_D = 4\pi p \; dx_1 \; dx_2 \; dx_3$$

$$\text{(viii)} \quad d\theta_B = 0$$

and the value of this form of Maxwell's equations is it does not involve use of the star operator. Additionally, if it were required to distinguish $\underline{H}$ from $\underline{B}$ and $\underline{D}$ from $\underline{E}$ then notice that two forms are used for $\underline{B}$, $\underline{D}$ and $\underline{J}$ whereas one forms correspond to $\underline{E}$ and $\underline{H}$. However, we still have for example,

$$\underline{D} \qquad \omega_D \qquad *\omega_D = \theta_D$$

In free space, the Maxwell equations simplify further because we have

$$\begin{array}{lcl} \underline{E} = \underline{D} & & \omega_E = \omega_D \\ \underline{H} = \underline{B} & \therefore & \omega_H = \omega_B \\ \underline{J} = 0 & & \omega_J = 0 \\ p = 0 & & \end{array}$$

and so equations (v)′ to (viii)′ become:

$$\left.\begin{aligned} d\omega_E &= -\frac{1}{C} *\dot{\omega}_H \\ d\omega_H &= \frac{1}{C} *\dot{\omega}_E \\ d(*\omega_E) &= 0 \\ d(*\omega_H) &= 0 \end{aligned}\right\}$$

Maxwell's equations in free space.

In the previous calculations, exterior derivation has been with respect to the space variables x_1, x_2, x_3 only, so the problem was assumed to be a three dimensional one. To contrast with this, take a four dimensional model based on coordinates (x_1, x_2, x_3, t) and use the Lorentz metric by choosing $\{dx_1, dx_2, dx_3, cdt\}$ as orthonomal basis with

$$\langle dx_i, dx_j \rangle = \delta_{ij}$$

$$\langle dx_i, cdt \rangle = 0$$

$$\langle cdt, cdt \rangle = -1$$

Use the results of page 61, namely

$$\left.\begin{aligned} &*(dx_j dx_k) = -c\, dx_i dt \\ \text{and}\quad &*(dx_i\, cdt) = dx_j dx_k \end{aligned}\right\} \qquad \ldots(A)$$

and construct approriate 2-forms.

The two Maxwell equations

$$\operatorname{curl} \underline{E} = -\frac{1}{C}\frac{\partial \underline{B}}{\partial t}$$

$$\operatorname{div} \underline{B} = 0$$

can be added to give a single equation which involves $\underline{B}$, $\underline{E}$ only.

This suggests that the 2-form

$$*\omega_B = B_1 dx_2 dx_3 + B_2 dx_3 dx_1 + B_3 dx_1 dx_2$$

$$= \sigma_B, \text{ say}$$

be combined with the 2-form

$$\omega_E c dt = \sigma_E, \text{ say,}$$

to give $\sigma = \sigma_E + \sigma_B$. Therefore

$$d\sigma = d\sigma_E + d\sigma_B =$$

$$[(\frac{\partial E_3}{\partial x_2} - \frac{\partial E_2}{\partial x_3}) dx_2 dx_3 + (\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1}) dx_3 dx_1 + (\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2}) dx_1 dx_2] c dt$$

$$+ (\frac{\partial B_1}{\partial t} dx_2 dx_3 + \frac{\partial B_2}{\partial t} dx_3 dx_1 + \frac{\partial B_3}{\partial t} dx_1 dx_2) dt + \operatorname{div} \underline{B}\, dx_1 dx_2 dx_3$$

$$= [\{(\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2}) + \frac{1}{C}\frac{\partial B_3}{\partial t}\} dx_1 dx_2 + \{(\frac{\partial E_3}{\partial x_2} - \frac{\partial E_2}{\partial x_3}) + \frac{1}{C}\frac{\partial B_1}{\partial t}\} dx_2 dx_3$$

$$+ \{(\frac{\partial E_1}{\partial x_3} - \frac{\partial E_2}{\partial x_1}) + \frac{1}{C}\frac{\partial B_2}{\partial t}\} dx_3 dx_1] c dt$$

$$+ \operatorname{div} \underline{B}\, dx_1 dx_2 dx_3$$

However, the Maxwell equations (i) and (iv) from page 103 are equivalent to the following set of equations:

$$\left.\begin{aligned}
\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2} + \frac{1}{C}\frac{\partial B_3}{\partial t} &= 0 \\
\frac{\partial E_3}{\partial x_2} - \frac{\partial E_2}{\partial x_3} + \frac{1}{C}\frac{\partial B_1}{\partial t} &= 0 \\
\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1} + \frac{1}{C}\frac{\partial B_2}{\partial t} &= 0
\end{aligned}\right\} \quad \underline{\text{and}}\ \operatorname{div} \underline{B} = 0.$$

and these are simply equivalent to the single equation

$$d\sigma = 0.$$

In free-space, $\underline{B} = \underline{H}$

$\therefore \quad \sigma_B = \sigma_H$

$\Rightarrow \quad \sigma = \sigma_E + \sigma_H$

$$\begin{aligned}
\therefore \quad *\sigma &= *(E_1 dx_1 dt + E_2 dx_2 dt + E_3 dx_3 dt)c \\
&\quad + *(H_1 dx_2 dx_3 + H_2 dx_3 dx_1 + H_3 dx_1 dx_2) \\
&= E_1 dx_2 dx_3 + E_2 dx_3 dx_1 + E_3 dx_1 dx_2 \\
&\qquad - (H_1 dx_1 + H_2 dx_2 + H_3 dx_3) c\,dt
\end{aligned}$$

by use of the equation $\underline{(A)}$ on page 106

$$\begin{aligned}
\therefore \quad d(*\sigma) &= \operatorname{div} \underline{E}\, dx_1 dx_2 dx_3 \\
&\quad + \frac{\partial E_1}{\partial t} dt dx_2 dx_3 + \frac{\partial E_2}{\partial t} dt dx_3 dx_1 + \frac{\partial E_3}{\partial t} dt dx_1 dx_2 \\
&\quad - \left[\left(\frac{\partial H_3}{\partial x_2} - \frac{\partial H_2}{\partial x_3}\right) dx_2 dx_3 + \left(\frac{\partial H_1}{\partial x_3} - \frac{\partial H_3}{\partial x_1}\right) dx_3 dx_1 + \left(\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial x_2}\right) dx_1 dx_2\right] c\,dt
\end{aligned}$$

$$= [\{\frac{1}{C}\frac{\partial E_3}{\partial t} - (\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial x_2})\}dx_1dx_2 + \{\frac{1}{C}\frac{\partial E_2}{\partial t} - (\frac{\partial H_1}{\partial x_3} - \frac{\partial H_3}{\partial x_1})\}dx_3dx_1$$

$$+ \{\frac{1}{C}\frac{\partial E_1}{\partial t} - (\frac{\partial H_3}{\partial x_2} - \frac{\partial H_2}{\partial x_3})\}dx_2dx_3]cdt$$

$$+ \text{div } \underline{E}\, dx_1dx_2dx_3$$

By examining coefficients it can be seen that the equations

$$\text{curl } \underline{H} = \frac{1}{C}\frac{\partial \underline{E}}{\partial t}$$

$$\text{div } \underline{E} = 0$$

are equivalent to the single equation

$$d(*\sigma) = 0.$$

Thus, in free space, the Maxwell equations are simply

$$\left.\begin{aligned} d\sigma &= 0 \\ d(*\sigma) &= 0 \end{aligned}\right\}$$ Maxwell's Equations in Free Space

where $\sigma = \sigma_E + \sigma_B$

$$= *\omega_B + \omega_E cdt$$

and where ω_B, ω_E are the dual 1-forms of the fields $\underline{B}$ and $\underline{E}$.

Flanders [2] introduces forms

$$\alpha = (E_1 dx_1 + E_2 dx_2 + E_3 dx_3) c dt + (B_1 dx_2 dx_3 + B_2 dx_3 dx_1 + B_3 dx_1 dx_2)$$

$$\beta = -(H_1 dx_1 + H_2 dx_2 + H_3 dx_3) c dt + (D_1 dx_2 dx_3 + D_2 dx_3 dx_1 + D_3 dx_1 dx_2)$$

$$\gamma = (J_1 dx_2 dx_3 + J_2 dx_3 dx_1 + J_3 dx_1 dx_2) dt - p dx_1 dx_2 dx_3$$

so that the full set of Maxwell Equations (I) to (iv) of page 103 become

$$d\alpha = 0 \qquad \ldots (1)$$

$$d\beta + 4\pi\gamma = 0 \qquad \ldots (2)$$

and the proof merely consists of checking, via equations (i) to (iv) that the coefficients in the forms

$d\alpha$ and $d\beta + 4\pi\gamma$ are zero.

3.1.3 Associated Results

Considering the Maxwell equations as given in the previous section will hopefully provide a closer illustration of the electromagnetic theory.

For example, (equation (viii) Page 104) gives

$$d(*\omega_B) = 0$$

In order to be able to apply the Poincare' Lemma, it is assumed that the region of space over which $*\omega_B$ is defined is contractible to a point and that physically this means there is an absence of magnetic monopoles. In this case $*\omega_B$ is exact and so

$$d\lambda = *\omega_B \text{ for some 1-form } \lambda.$$

$$\therefore \quad \omega_B = *d\lambda = \text{curl } \lambda$$

Classically, this is written as

$$\text{curl } \underline{A} = \underline{B}$$

where $\underline{A}$ is the <u>vector potential</u>.

However, λ is not unique because for 0-form ϕ we have

$$d\lambda = d(\lambda+d\phi) = d(\lambda+\text{grad } \phi)$$

meaning that λ is only determined up to the gradient of a 0-form.

Considering the Maxwell equations given on page 109 (quoted from Flanders) note that

$$d\beta + 4\pi\gamma = 0$$

$$\therefore \quad d\gamma = 0$$

Combining this last equation with Stoke's theorem applied to a space-time region, R, bounded by the two spacelike 3-surfaces V_1, V_2 which form the boundary R we get

$$0 = \int_R d\gamma = \int_{\partial R} \gamma = \int_{V_1}^{V_2} p\,dx_1 dx_2 dx_3$$

since dt = 0 over this latter integral

$$\therefore \quad 0 = \int_{V_1}^{V_2} p\,dx_1 dx_2 dx_3 = Q(V_2) - Q(V_1)$$

which is the law of the conservation of electric charge.

3.2.1 Superconductivity and the London Equations

Some metals, for example mercury, when cooled to temperatures of around absolute zero behave like perfect conductors and are known as superconductors. One theory which describes this phenomenon is known as the London Theory and is due to F. London and H. London. This theory suggests that the current $\underline{J}$ in the superconductor consists of a supercurrent j and a normal current $\underline{j}_n$.

The superconducting electrons, when influenced by a field $\underline{E}$, have the following equation or motion:

$$m \frac{d\underline{v}}{dt} = e\,\underline{E} \quad \ldots\ldots (1)$$

where $\underline{v}$ is their velocity, t is time and e and m are the electron charge and mass respectively. Hence,

$$\underline{j}_s = n_s e\,\underline{v} \quad \ldots\ldots (2)$$

where n_s is the number of superconducting electrons per unit volume. Equations (1) and (2) give

$$\frac{m}{n_s e^2} \frac{\partial \underline{j}_s}{\partial t} = \underline{E} \quad \ldots\ldots (3)$$

But, from Maxwell's equations, we have

$$\text{curl}\ \underline{E} = -\frac{1}{C} \frac{\partial \underline{H}}{\partial t}$$

so that equation (3) becomes

$$\text{curl}\left[\left(\frac{mc}{n_s e^2}\right)\frac{\partial \underline{j_s}}{\partial t}\right] = -\frac{\partial \underline{H}}{\partial t}$$

Integrating with respect to time we get

$$\text{curl}\left[\left(\frac{mc}{n_s e^2}\right)\underline{j_s}\right] = -\underline{H} + \text{constant.}$$

The London theory assumes this constant to be zero. Hence we have

$$\text{curl}\,(\Lambda\,\underline{j_s}) = -\underline{H} \;\ldots\ldots\; (4)$$

where $\Lambda = \frac{mc}{n_s e^2}$ is constant.

Collecting together the equations of the previous page we have:

Eqn. (4) is $\quad \text{curl}\,(\Lambda\,\underline{j_s}) = -\underline{H} \;\ldots\ldots(i)$

Eqn. (3) is $\quad \frac{\partial}{\partial t}(\Lambda\,\underline{j_s}) = \underline{E} \;\ldots\ldots(ii)$

By definition, $\quad \underline{J} = \underline{j_s} + \underline{j_n} \;\ldots\ldots\; (iii)$

For the normal current, $\quad \underline{j_n} = \sigma\,\underline{E} \;\ldots\ldots\; (iv)$

Also, the Maxwell equations are

$$\text{curl}\,\underline{H} = \frac{4\pi}{C}\underline{J} + \frac{1}{C}\frac{\partial \underline{E}}{\partial t} \;\ldots\ldots\; (v)$$

$$\text{curl}\,\underline{E} = -\frac{1}{C}\frac{\partial \underline{H}}{\partial t} \;\ldots\ldots\; (vi)$$

$$\text{div}\,\underline{H} = 0 \;\ldots\ldots\; (vii)$$

$$\text{div}\,\underline{E} = 4\pi p \;\ldots\ldots\; (viii)$$

Equations (i) to (iv) embody the additional assumptions of the London equations.

The aim of this section is to express these equations in terms of differential forms. To do this, note that the fields $\underline{E}$, $\underline{H}$ and $\underline{J}$ have dual 1-forms ω_E, ω_H and ω_J respectively.

Equations (iii) and (iv) are not differential relationships, so write

$$\underline{j}_s = \underline{J} - \underline{j}_n = \underline{J} - \sigma\,\underline{E}$$

so equations (i) to (iv) reduce to

$$\Lambda \text{ curl } (\underline{J} - \sigma\,\underline{E}) = -\frac{1}{C}\,\underline{H} \quad \ldots\ldots\text{(ix)}$$

$$\Lambda \frac{\partial}{\partial t}(\underline{J} - \sigma\,\underline{E}) = \underline{E} \quad \ldots\ldots\ldots\ \text{(x)}$$

Equation (ix) above can be written

$$\Lambda{*}d(\omega_J - \sigma\omega_E) = -\frac{1}{C}\,\omega_H$$

i.e. $$\Lambda d(\omega_J - \sigma\omega_E) = -\frac{1}{C}\,{*}\omega_H$$

Equation (x) is merely

$$\Lambda(\dot{\omega}_J - \sigma\dot{\omega}_E) = \omega_E$$

Giving equations (v) to (viii) of the Maxwell equations in the form quoted on page 103 together with the above two equations gives

$$d\omega_E = -\frac{1}{C}\,{*}\dot{\omega}_H \qquad \text{... (xi)}$$

$$d\omega_H = \frac{4\pi}{C}\,{*}\omega_J + \frac{1}{C}\,{*}\dot{\omega}_E \qquad \text{... (xii)}$$

$${*}d({*}\omega_E) = 4\pi p \qquad \text{... (xiii)}$$

$$d({*}\omega_H) = 0 \qquad \text{... (xiv)}$$

$$d(\omega_J - \sigma\omega_E) = -\frac{1}{C\Lambda}\,{*}\omega_H \qquad \text{... (xv)}$$

$$\dot{\omega}_J - \sigma\dot{\omega}_E = \frac{1}{\Lambda}\,\omega_E \qquad \text{... (xvi)}$$

In view of the property $d^2 = 0$ and by using equation (xv) above, equation (xiv) is redundant. Also, if we put

$$\omega_{j_s} = \omega_J - \sigma\omega_E$$

the above equations contract to

$$\left.\begin{aligned} d\omega_E &= -\frac{1}{C} *\dot{\omega}_H \\ d\omega_H &= \frac{4\pi}{C} *\omega_J + \frac{1}{C} *\dot{\omega}_E \\ d(*\omega_E) &= *4\pi p \\ d\omega_{j_s} &= -\frac{1}{C\Lambda} *\omega_H \\ \dot{\omega}_{j_s} &= \frac{1}{\Lambda}\,\omega_E \end{aligned}\right\} \quad \text{London Equations}$$

By regarding fields $\underline{E}$, $\underline{H}$, $\underline{J}$ and the charge density p as being functions of x_1, x_2, x_3 and t, the problem becomes 4-dimensional. Consider the forms

$$\alpha_1 = (E_1 dx_1 + E_2 dx_2 + E_3 dx_3) cdt$$

$$\alpha_2 = H_1 dx_2 dx_3 + H_2 dx_3 dx_1 + H_3 dx_1 dx_2$$

$$\beta_1 = -\frac{1}{C}(J_1 dx_1 + J_2 dx_2 + J_3 dx_3)$$

$$\beta_2 = p\ cdt.$$

With the Lorentz metric on 4-space, the basis

$$(dx_1, dx_2, dx_3, cdt)$$

is orthonormal. Use also,

$$\left.\begin{aligned} *(dx_i dx_j) &= -dx_k cdt \\ *(dx_i cdt &= dx_j dx_k \end{aligned}\right\} \quad \text{i,j,k in cyclic order.}$$

and

$$*dx_1 = -dx_2dx_3cdt$$

$$*dx_2 = -dx_3dx_1cdt$$

$$*dx_3 = -\,dx_1dx_2cdt$$

$$*cdt = -\,dx_1dx_2dx_3$$

These give

$$*\alpha_1 = E_1dx_2dx_3 + E_2dx_3dx_1 + E_3dx_1dx_2$$

$$*\alpha_2 = -(H_1dx_1 + H_2dx_2 + H_3dx_3)cdt$$

$$*\beta_1 = (J_1dx_2dx_3 + J_2dx_3dx_1 + J_3dx_1dx_2)dt$$

$$*\beta_2 = -pdx_1dx_2dx_3$$

With the above forms in mind, the equations (vi) and (vii) of page 113 can be written in the form

$$d(\alpha_1 + \alpha_2) = 0.$$

Proof

$$d(\alpha_1 + \alpha_2) = (\frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial x_2} + \frac{\partial H_3}{\partial x_3})dx_1dx_2dx_3$$

$$+ [\{(\frac{\partial E_3}{\partial x_2} - \frac{\partial E_2}{\partial x_3}) + \frac{1}{C}\frac{\partial H_1}{\partial t}\}dx_2dx_3 + \{(\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1}) + \frac{1}{C}\frac{\partial H_2}{\partial t}\}dx_3dx_1$$

$$+ \{(\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2}) + \frac{1}{C}\frac{\partial H_3}{\partial t}\}dx_1dx_2]$$

if this is zero then

$$\frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial x_2} + \frac{\partial H_3}{\partial x_3} = 0 \qquad \underline{\text{i.e.}}\ \text{div}\ \underline{H} = 0$$

and

$$\frac{\partial E_3}{\partial x_2} - \frac{\partial E_2}{\partial x_3} = -\frac{1}{C}\frac{\partial H_1}{\partial t} \qquad \text{etc.}$$

i.e. $\quad \text{curl}\ \underline{E} = -\frac{1}{C}\frac{\partial \underline{H}}{\partial t}$

Also, equations (v) and (viii) of the London equations may be written as

$$d^*(\alpha_1+\alpha_2) + 4\pi^*(\beta_1+\beta_2) = 0$$

Proof

$$*(\alpha_1+\alpha_2) = (E_1dx_2dx_3 + E_2dx_3dx_1 + E_3dx_1dx_2) -$$

$$- (H_1dx_1 + H_2dx_2 + H_3dx_3)cdt$$

$$\therefore \quad d*(\alpha_1+\alpha_2) =$$

$$- [\{(\frac{\partial H_3}{\partial x_2} - \frac{\partial H_2}{\partial x_3}) - \frac{1}{C}\frac{\partial E_1}{\partial t}\}dx_2dx_3 + \{(\frac{\partial H_3}{\partial x_1} - \frac{\partial H_1}{\partial x_3}) - \frac{1}{C}\frac{\partial E_2}{\partial t}\}dx_3dx_1$$

$$+ \{(\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial x_2}) - \frac{1}{C}\frac{\partial E_3}{\partial t}\}dx_1dx_2]cdt + \text{div } \underline{E}\ dx_1dx_2dx_3$$

Also,

$$*(\beta_1+\beta_2) = \frac{1}{C}\{J_1dx_2dx_3 + J_2dx_3dx_1 + J_3dx_1dx_2\}cdt$$

$$- pdx_1dx_2dx_3$$

Hence,

$$d*(\alpha_1+\alpha_2) + 4\pi*(\beta_1+\beta_2) = 0$$

$$\Leftrightarrow \text{div } \underline{E} = 4\pi p \qquad \text{(equation (viii))}$$

and

$$\frac{\partial H_3}{\partial x_2} - \frac{\partial H_2}{\partial x_3} = \frac{4\pi}{C}J_1 + \frac{1}{C}\frac{\partial E_1}{\partial t} \quad \text{etc, etc}$$

to give also,

$$\text{curl } \underline{H} = \frac{4\pi}{C}\underline{J} + \frac{1}{C}\frac{\partial \underline{E}}{\partial t}$$

which is equation (v).

There are two more of the London equations to be considered, namely equations (ix), (x) from page 114

<u>i.e.</u> $\operatorname{curl}(\underline{J} - \sigma\underline{E}) = \frac{-1}{C\Lambda}\underline{H}$

$$\Lambda \frac{\partial}{\partial t}(\underline{J} - \sigma\underline{E}) = \underline{E}$$

The first of these is equivalent to

$$\sigma d\alpha_1 = (\tfrac{1}{\Lambda}\alpha_2 - c^2 d\beta_1)dt$$

whilst the second is equivalent to

$$\sigma[d{*}\alpha_1 + 4\pi{*}\beta_2] = {*}(\tfrac{1}{C}\alpha_1 + d\beta_1)cdt$$

The proof is a tedious, but straight-forward process of equating coefficients. Summarizing, the London equations are equivalent to

$$\left.\begin{aligned} & d(\alpha_1+\alpha_2) = 0 \\ & d{*}(\alpha_1+\alpha_2) + 4\pi{*}(\beta_1+\beta_2) = 0 \\ & \sigma d\alpha_1 = (\tfrac{1}{\Lambda}\alpha_2 - c^2 d\beta_1)dt \\ & \sigma[d({*}\alpha_1) + 4\pi{*}\beta_2] = {*}(\tfrac{1}{C}\alpha_1 + d\beta_1)cdt \end{aligned}\right\}$$

3.3.1 Further Developments

The aim of this section is to give some indication of how the methods of this and the previous chapter can be further developed. The core idea hitherto has been that of a differential form on a manifold and there are two branches of mathematics which themselves have this device as a foundation stone. These branches are De Rham cohomology and a modern treatment of differential geometry. For some time, both these topics have been applied to the solution of problems in physics in general and electromagnetic theory in particular. To give some idea of their applicability, the relevant ideas will be briefly outlined.

Let M be a C^∞ manifold and denote by

$$Z^p(M)$$

the vector space of closed p-forms on M. Let

$$B^p(M)$$

denote the vector space of exact p-forms on M. By virtue of the property

$$d^2 = 0$$

for exterior derivative d, we have

$$B^p(M) \subset Z^p(M)$$

The quotient

$$H^p(M) = Z^p(M)/B^p(M)$$

is called the p^{th} De Rham cohomology group. Its cosets are actually equivalence classes of $Z^p(M)$ with respect to the relation

$\alpha \sim \beta$ iff α,β differ by an exact form.

This defines, for example, vector potential as being an equivalence class of α-forms (see page 111). It can be shown that these groups are topological invariants and provide information about the topological characteristics of the under-lying manifold - particularly in the case of connected, orientable manifolds of low dimension such as surfaces in 3-space or 3-manifolds in space-time.

For example, for a connected compact orientable 2-manifold embedded in $\mathbb{R}^3$ is homeomorphic to a sphere with a number of handles on it. The number of handles is simply given by

$$\frac{1}{2} \dim H'(M).$$

However, as mentioned in Section 3.1.3, if a region of space contains a magnetic charge, then this would appear as a hole in the underlying manifold. In terms of the 1-forms, when there are no holes, any closed 1-form is exact.

It is at this level that one begins to see the relevance of cohomology to electromagnetic theory but a further description would involve ideas which are beyond the scope of this dissertation. One reference is, however, Roberts [5].

Modern differential geometry, as described by Hicks [4], generalizes many of the results which are applicable to surfaces and curves in $\mathbb{R}^3$ to Riemannian n-manifolds.

This means that problems in 4-space may be approached, for example, by these techniques.

The connection between differential geometry and electromagnetic theory are made evident by the sort of examples quoted below:

(a) Charge density on a conductor is greatest where the surface is sharply convex. Generally, therefore, there is a strong relationship between the curvature of a surface and the charge density on it.

(b) The path of a particle in 4-space is geodesic when electromagnetic fields may also be exerting their influence.

Differential geometry, therefore, seems to be highly relevant to this type of analysis.

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